Reactive Synthesis

Lecture 7

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(Saarland University)

Plan for Today

Have fun!

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My kind of fun:

- Infinite games in infinite arenas
- An undetermined game

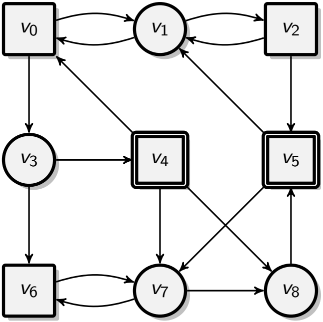
Recap: Attractors

- The *controlled predecessor* $\mathrm{CPre}_0(R)$ of R is defined as

$$\mathrm{CPre}_0(R) = \{v \in V_0 \mid v' \in R \text{ for some successor } v' \text{ of } v\} \cup \{v \in V_1 \mid v' \in R \text{ for all successors } v' \text{ of } v\}.$$

- The *(Player 0) attractor* $\mathrm{Attr}_0(R)$ of R is defined as
  - $\mathrm{Attr}_0^0(R) = R$,
  - $\mathrm{Attr}_0^{n+1}(R) = \mathrm{Attr}_0^n(R) \cup \mathrm{CPre}_0(\mathrm{Attr}_0^n(R))$, and
  - $\mathrm{Attr}_0(R) = \bigcup_{n \in \mathbb{N}} \mathrm{Attr}_0^n(R)$.

Recap: Attractors



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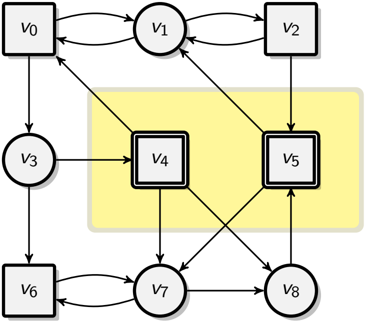

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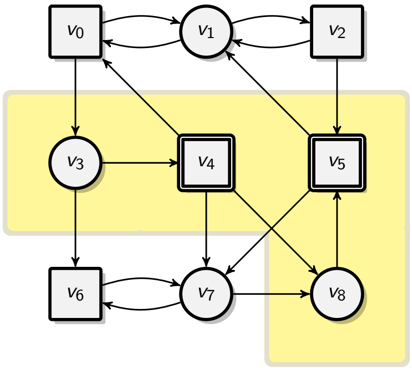

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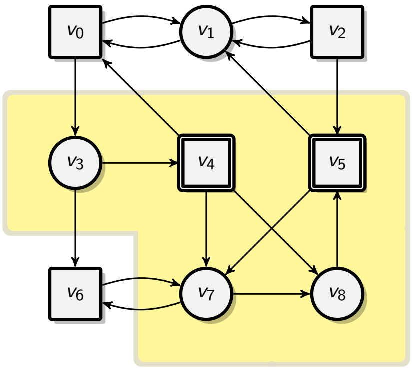

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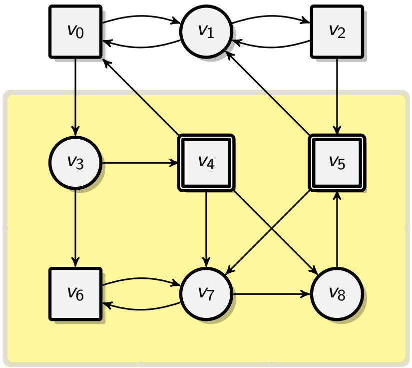
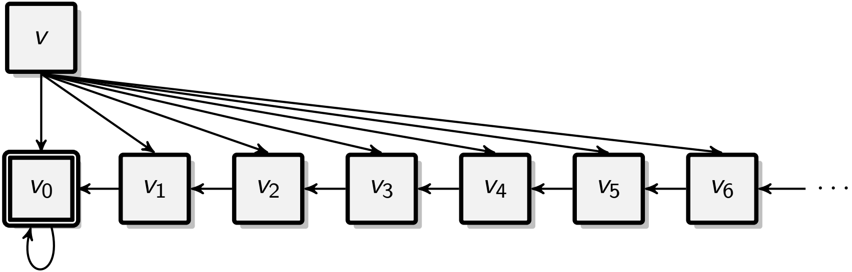


A Problem



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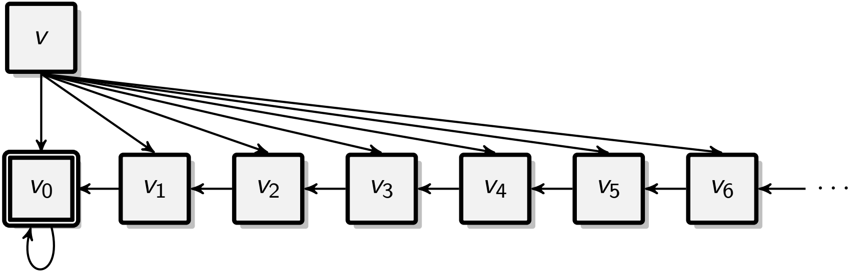

- We have $\mathrm{Attr}_0^n(\{v_0\}) = \{v_j \mid j \leq n\}$ for every $n \in \mathbb{N}$.
- Hence, $v \notin \mathrm{Attr}_0(\{v_0\}) = \bigcup_{n \in \mathbb{N}} \mathrm{Attr}_0^n(\{v_0\}) = \{v_j \mid j \in \mathbb{N}\}$.
- But, v is in Player 0's winning region.

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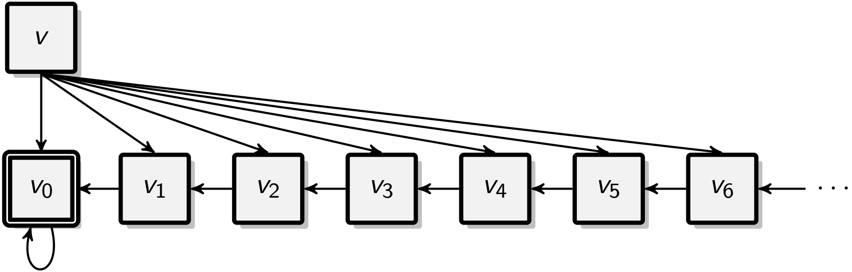

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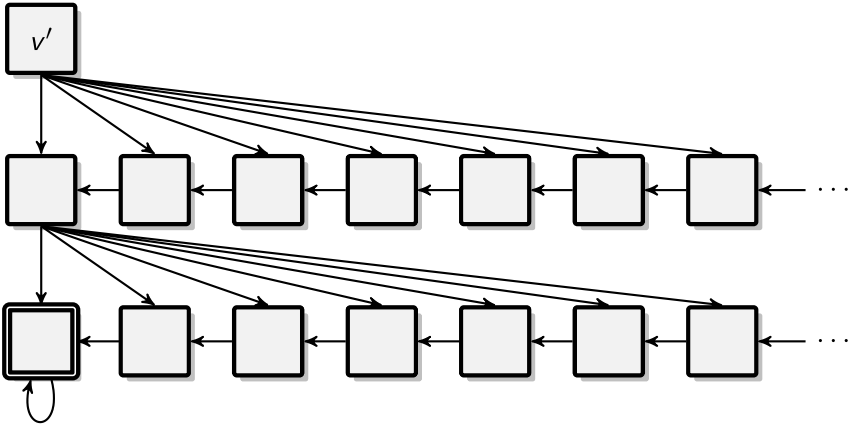
- But, v is in Player 0's winning region.
- Solution: apply CPre_0 to $\bigcup_{n\in\mathbb{N}} \mathrm{Attr}_0^n(\{v_0\})$.

Pushing It Further



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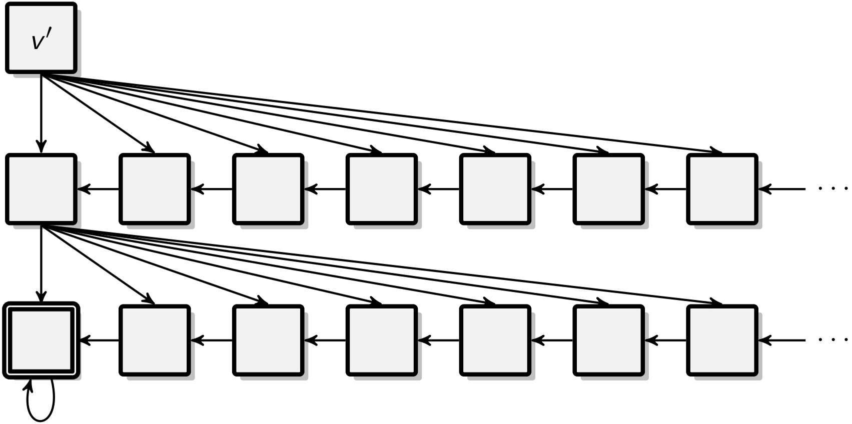
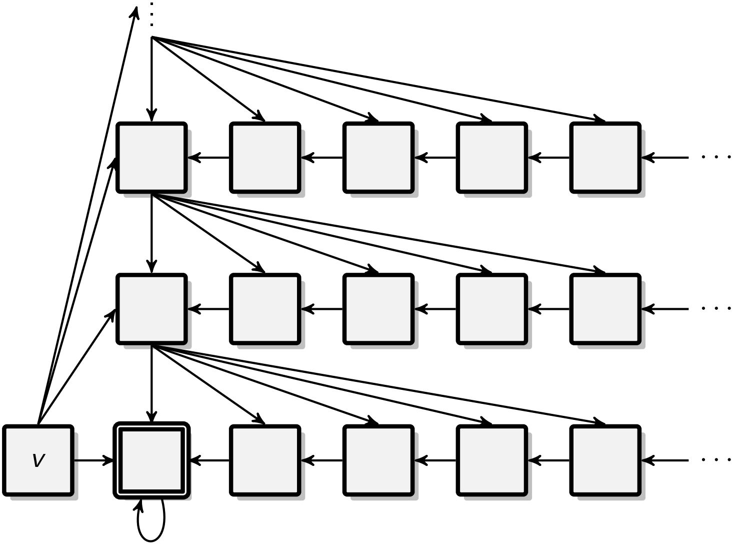


- Applying CPre_0 to $\bigcup_{n\in\mathbb{N}} \mathrm{Attr}_0^n(\{v_0\})$ repeatedly adds all vertices in the second row (one in each stage).. but not v'.
- Thus, again take the union over all smaller stages and then apply CPre_0 to add v'.

And Even Further



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Note

1. $n + 1 = \{0, 1, \ldots, n\}$, i.e., set membership $\in$ coincides with the natural order $\leq$ (which is a well-order).
2. Every element of n is also a subset of n.

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- $\mathbb{Q}$ can be well-ordered, e.g., using a bijection to $\mathbb{N}$.
- What about well-ordering $\mathbb{R}$?

von Neumann Ordinals

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A set S is an ordinal if, and only if, it is well-ordered by $\in$ and if every element of S is also a subset of S.

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- $\omega + \omega = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}$ is an ordinal, typically denoted as $\omega \cdot 2$.

Back to Attractors

There are three types of ordinals:

1. $0 = \emptyset$.
2. Successor ordinals of the form $\alpha \cup \{\alpha\}$, denoted as $\alpha + 1$.
3. Limit ordinals, i.e., those that are not zero and not successor ordinals, e.g., ω and $\omega \cdot 2$.

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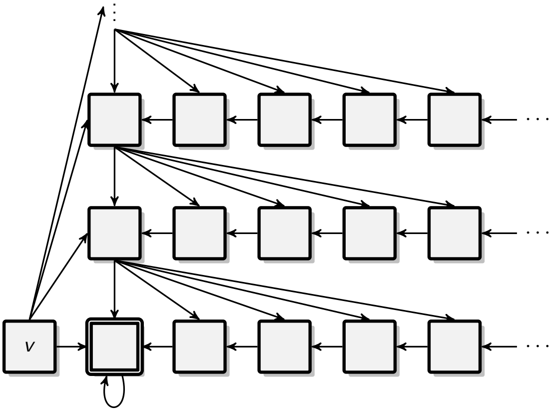
Given a (possibly infinite) arena $\mathcal{A} = (V, V_0, V_1, E)$ and $R \subseteq V$ define

- $\mathrm{Attr}_0^0(R) = R$,
- $\mathrm{Attr}_0^{\alpha+1}(R) = \mathrm{Attr}_0^{\alpha}(R) \cup \mathrm{CPre}_0(\mathrm{Attr}_0^{\alpha}(R))$ for a successor ordinal $\alpha + 1$, and
- $\mathrm{Attr}_0^{\alpha}(R) = \bigcup_{\alpha' < \alpha} \mathrm{Attr}_0^{\alpha'}(R)$ for a limit ordinal α.

Back to the Example



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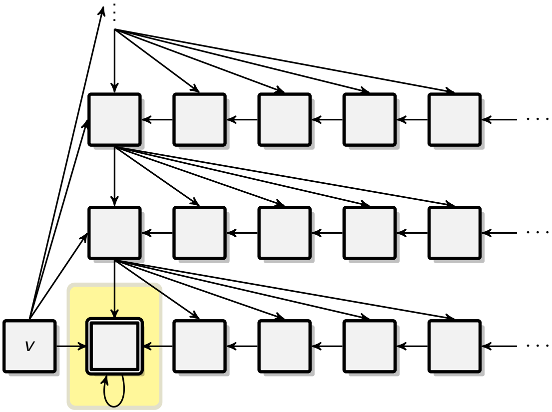

$$\mathrm{Attr}_0^0(\{v_0\})$$

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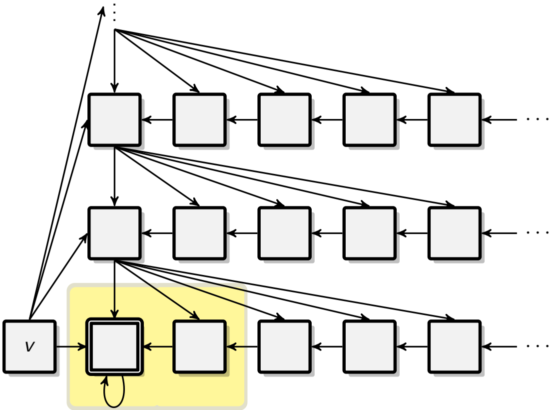

$$\mathrm{Attr}_0^1(\{v_0\})$$

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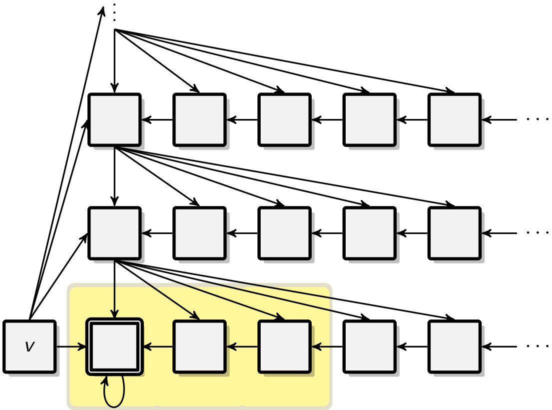

$$\mathrm{Attr}_0^2(\{v_0\})$$

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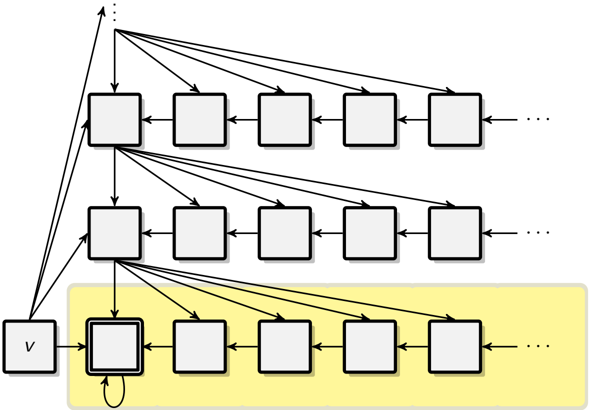

$$\mathrm{Attr}_0^{\omega}(\{v_0\})$$

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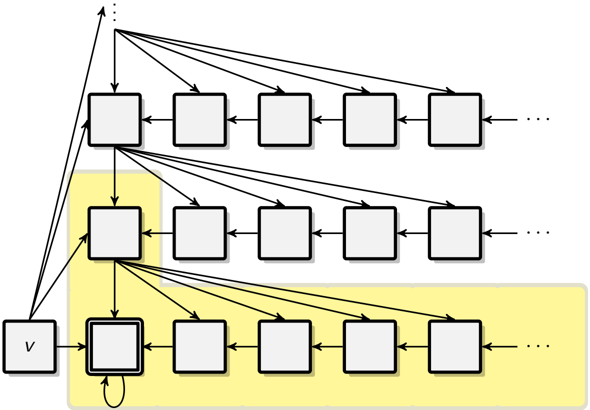

$$\mathrm{Attr}_0^{\omega+1}(\{v_0\})$$

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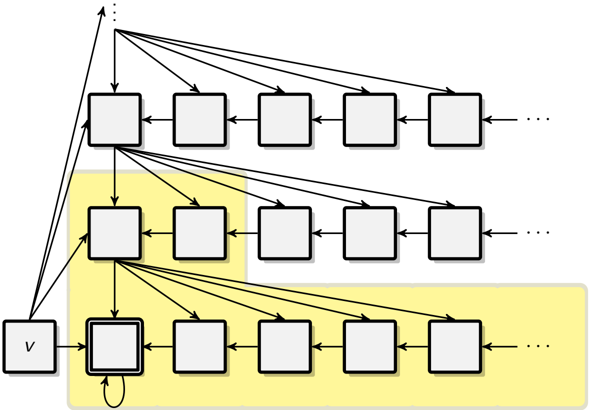
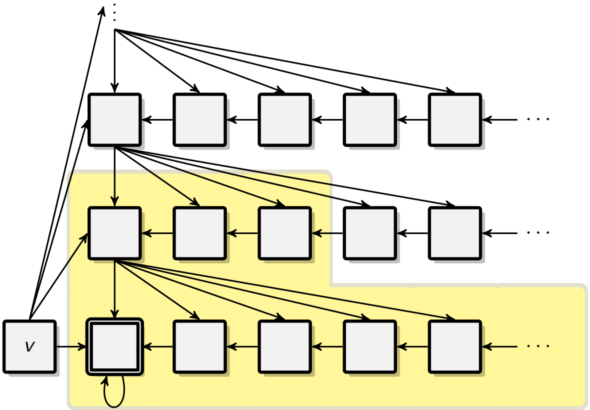


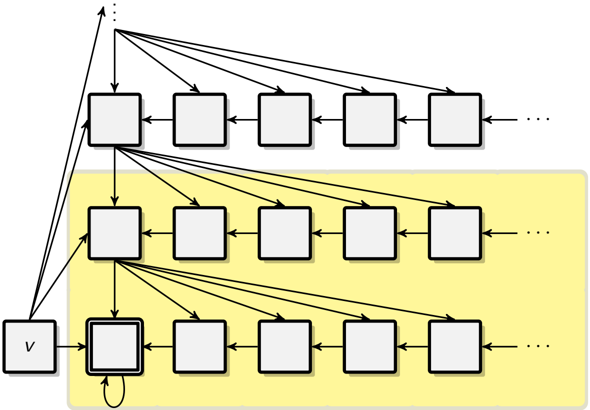
$$\mathrm{Attr}_0^{\omega+2}(\{v_0\})$$

Back to the Example

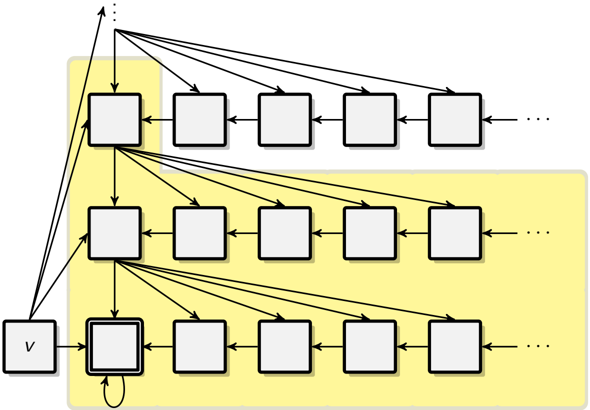


$$\mathrm{Attr}_0^{\omega+3}(\{v_0\})$$

Back to the Example



$$\mathrm{Attr}_0^{\omega\cdot 2}(\{v_0\})$$



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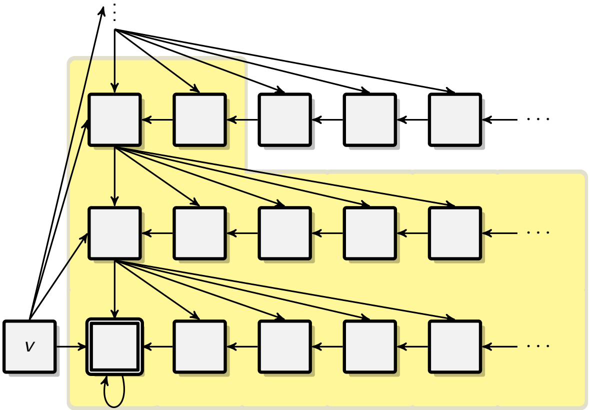
Back to the Example



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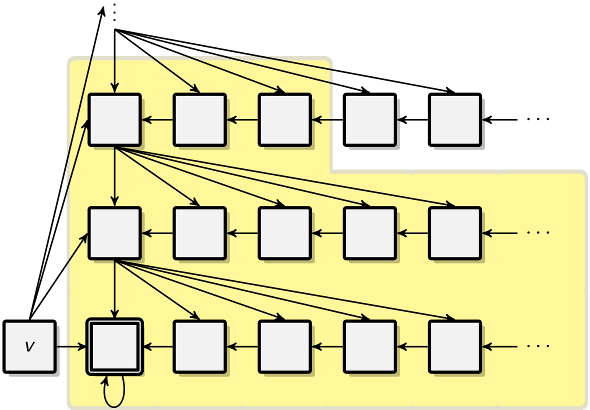

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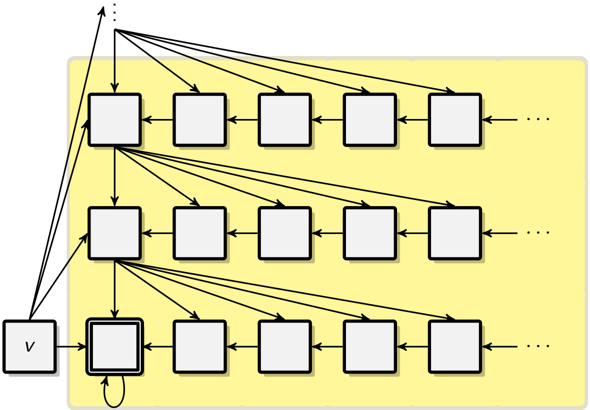

$$\mathrm{Attr}_0^{\omega\cdot 3}(\{v_0\})$$

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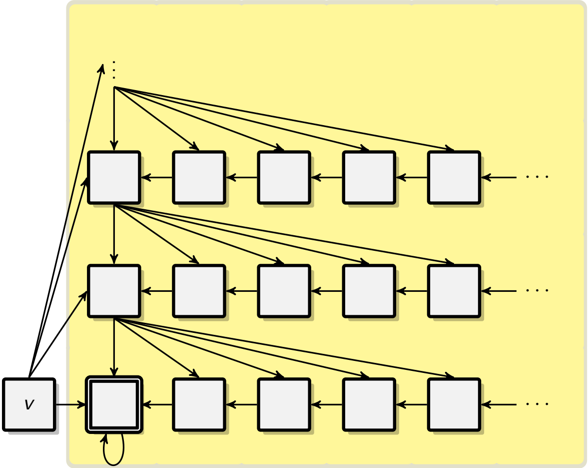

$$\mathrm{Attr}_0^{\omega^2}(\{v_0\})$$

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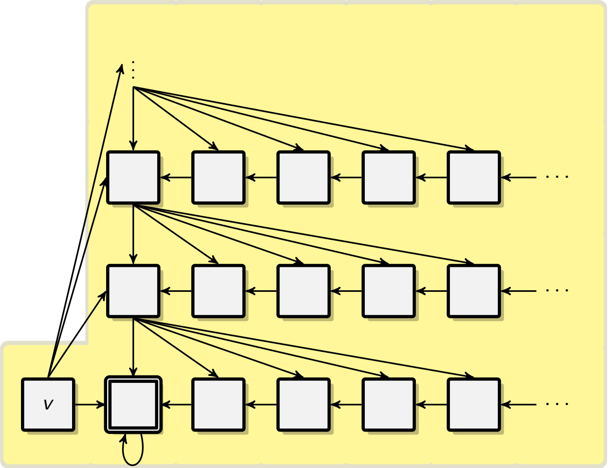

$$\mathrm{Attr}_0^{\omega^2+1}(\{v_0\})$$

Theorem

For every (possibly infinite) reachability game $\mathcal{G} = (\mathcal{A}, \textsc{Reach}(R))$*, there is an ordinal* α *such that*

$$\mathrm{Attr}_0^{\alpha}(R) = W_0(\mathcal{G}).$$

Have fun!

My kind of fun:

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- An undetermined game

Chomp

- There is a (rectangular) chocolate bar with $m \times n$ pieces.
- A move consists of taking a piece and all others that are to the right and above.
- The players move in alternation, starting with Player 0.
- The player who takes the bottom-left piece loses.
- This description can be formalized as a reachability game.

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Milka
PLAYER 0'S TURN

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Note

- The proof is non-constructive..
- .. winning strategy only known for special cases $n \times n$, $2 \times n$, and $n \times 2$ (try to find them).

Hamming Distance

In the following: $\mathbb{B} = \{0, 1\}$

Definition

For $x = x_0 x_1 x_2 \cdots$ and $y = y_0 y_1 y_2 \cdots$ in $\mathbb{B}^\omega$, the *Hamming distance* between x and y is defined as

$$\mathrm{hd}(x, y) = |\{n \in \mathbb{N} \mid x_n \neq y_n\}| \in \mathbb{N} \cup \{\infty\}.$$

Example

- $\mathrm{hd}(0101101000 \cdots,$
  $1010100000 \cdots) = 5$
- $\mathrm{hd}(1010101010 \cdots,$
  $0101010101 \cdots) = \infty$
- $\mathrm{hd}(1010101010 \cdots,$
  $1111111111 \cdots) = \infty.$

Definition

A function $f\colon \mathbb{B}^\omega \to \mathbb{B}$ is an *infinite XOR function*, if $hd(x, y) = 1$ implies $f(x) \neq f(y)$ for all $x, y \in \mathbb{B}^\omega$.

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I have none.. we will come back to this later.

Infinite XOR Functions

Definition

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Example

I have none.. we will come back to this later.

Theorem

There exists an infinite XOR function.

Proof

Define $x \sim y$ if $\mathrm{hd}(x, y) < \infty$.

Remark

$x \sim y \Leftrightarrow$ there are $n \in \mathbb{N}$ and $z \in \mathbb{B}^\omega$ with

$$x = x_0 \cdots x_n z \qquad \textit{and} \qquad y = y_0 \cdots y_n z.$$

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Lemma

$\sim$ is an equivalence relation.

Proof.

1. Reflexitivity: trivial, as $\mathrm{hd}(x, x) = 0 < \infty$.
2. Symmetry: trivial, as $\mathrm{hd}(x, y) = \mathrm{hd}(y, x)$.
3. Transitivity: apply the previous remark. □

Proof

- Let $S \subseteq \mathbb{B}^\omega$ be a set that contains exactly one element from each $\sim$-equivalence class.
- Hence, for every $x \in \mathbb{B}^\omega$, let $r(x)$ denote the unique element in S with $x \sim r(x)$.
- Define $f(x) = \mathrm{hd}(x, r(x))$ mod 2.
- This is well-defined, since $\mathrm{hd}(x, r(x)) \in \mathbb{N}$ for all $x \in \mathbb{B}^\omega$.

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Claim: f is an infinite XOR function.

Proof

Fix $x = x_0 x_1 x_2 \cdots$ and $y = y_0 y_1 y_2 \cdots$ with $\mathrm{hd}(x, y) = 1$.

- W.l.o.g., $x = w0z$ and $y = w1z$ for some $w \in \mathbb{B}^*$ and $z \in \mathbb{B}^\omega$.
- We have $x \sim y$ and thus $r(x) = r(y) =: r = r_0 r_1 r_2 \cdots$.

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- Let $D_x = \{n \mid x_n \neq r_n\}$ and $D_y = \{n \mid y_n \neq r_n\}$.
- $D_x \setminus \{|w|\} = D_y \setminus \{|w|\}$, as x and y only differ at position $|w|$.

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Hence,

$$f(x) = \mathrm{hd}(x, r) \bmod 2 = |D_x| \bmod 2$$

and

$$f(y) = \mathrm{hd}(y, r) \bmod 2 = |D_y| \bmod 2 = (|D_x| + 1) \bmod 2,$$

i.e., $f(x) \neq f(y)$.

The Game $\mathcal{G}_f$

- Fix some infinite XOR function f.
- We define a game $\mathcal{G}_f$ between Player 0 and Player 1. For the sake of simplicity, it is not played in an arena, but the players pick sequences of bits in alternation.

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- Formally, $\mathcal{G}_f$ is played in rounds $n = 0, 1, 2, \ldots$.
- In round n, first Player 0 picks $w_{2n} \in \mathbb{B}^+$, then Player 1 picks $w_{2n+1} \in \mathbb{B}^+$.
- Play $w_0, w_1, w_2, \ldots$ is won by Player $f(w_0 w_1 w_2 \cdots)$.

There are Undetermined Games

Theorem

Let f be an infinite XOR function. No player has a winning strategy for $\mathcal{G}_f$.

Proof Idea

Strategy stealing:

- For every strategy τ of Player 1, we construct two counter strategies σ and σ' that mimic τ.
- The only difference between σ and σ' is that one starts by playing a 0, the other by playing a 1.
- The remainder of the plays resulting from playing σ and σ' against τ are equal.
- Hence, their Hamming distance is 1 and one of the plays is won by Player 0.
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- Thus, τ is not a winning strategy.

The argument showing that Player 0 has no winning strategy is similar.

Proof

Let τ be a strategy for Player 1 in $\mathcal{G}_f$. We show that τ is not winning by constructing counter strategies σ and σ' as above.

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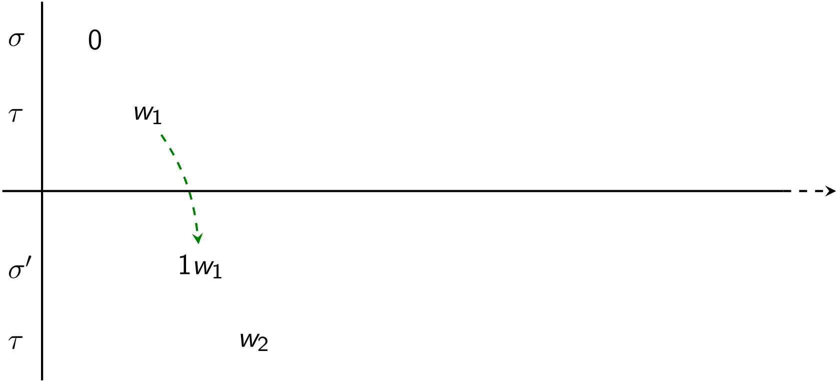

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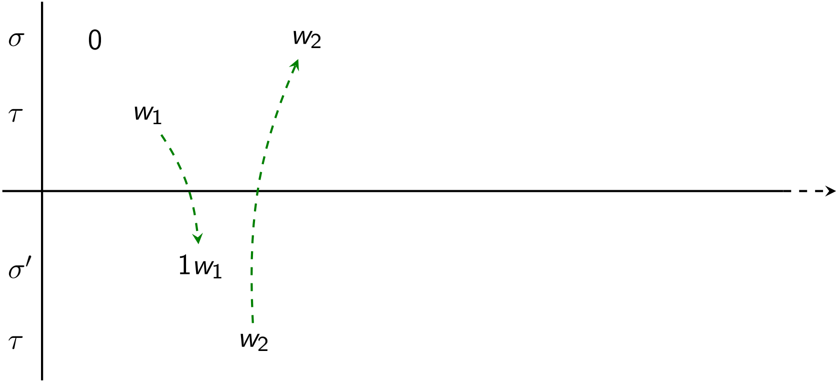

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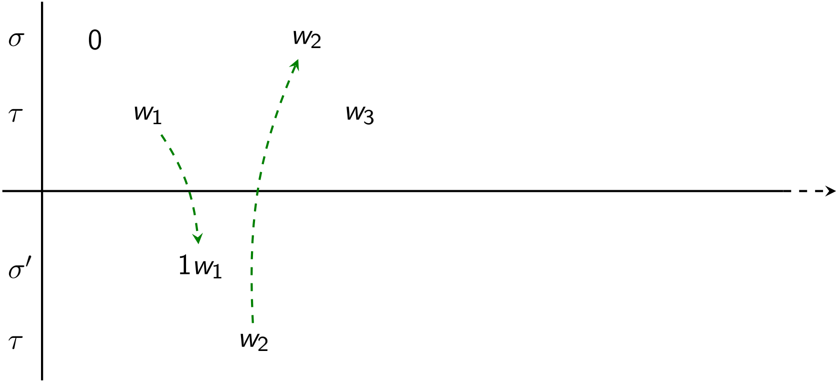

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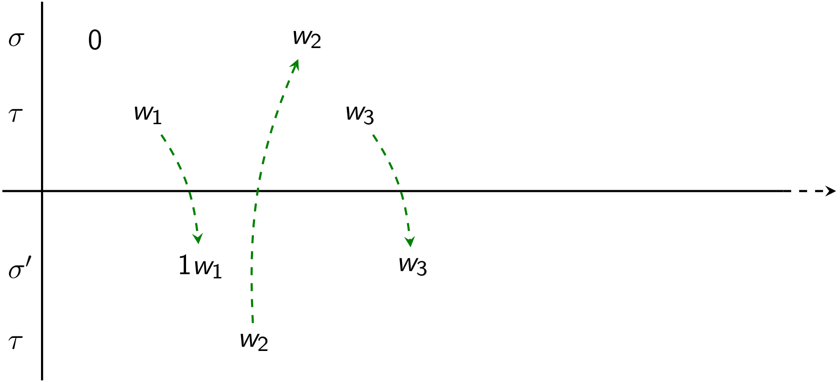

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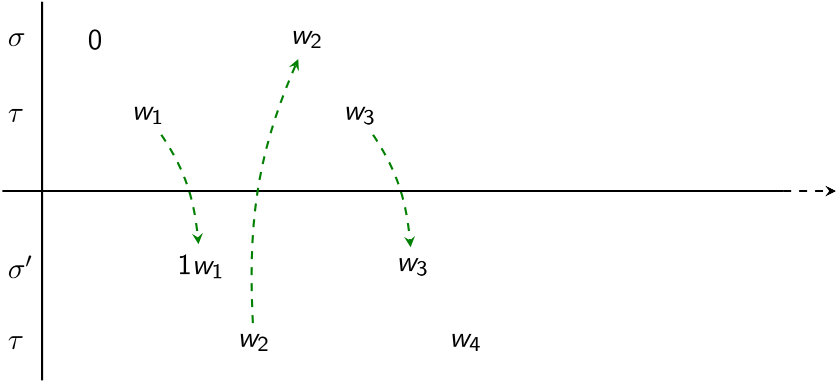

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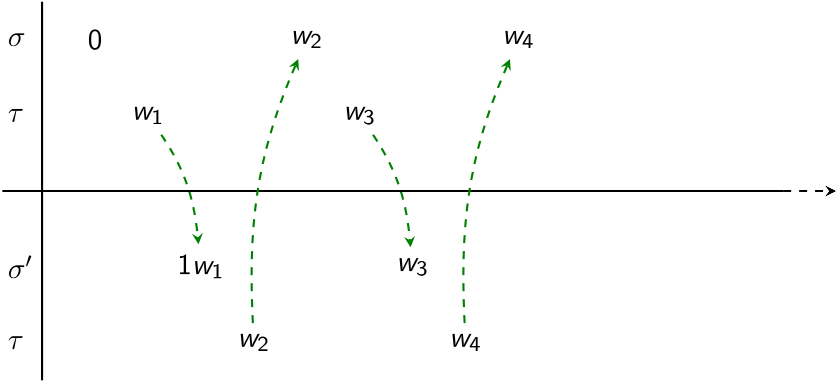

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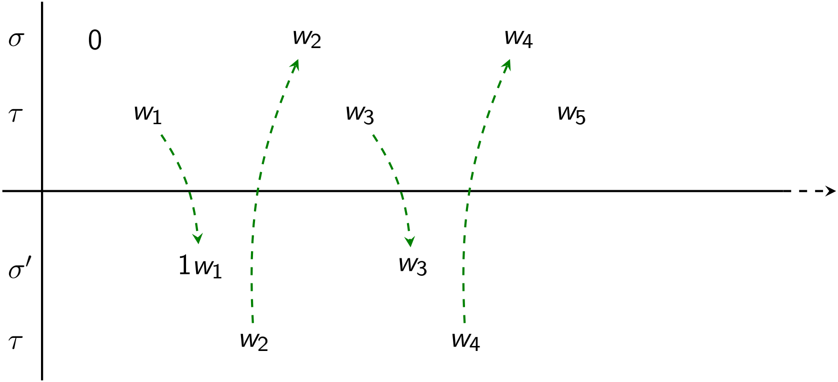

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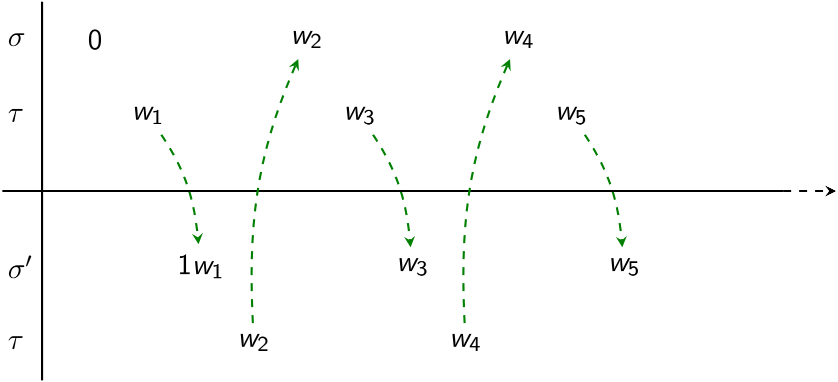

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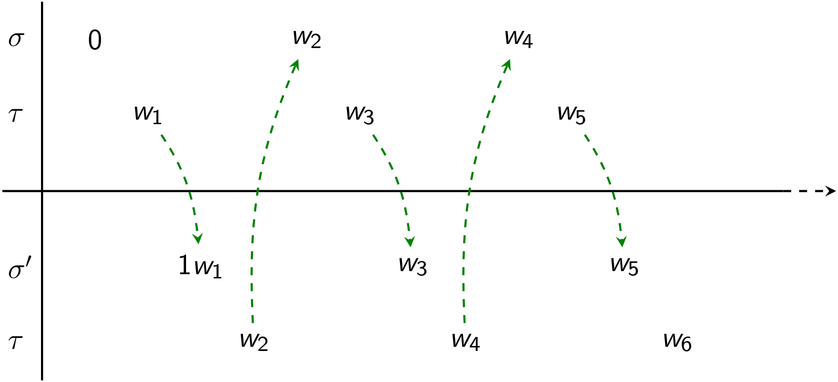

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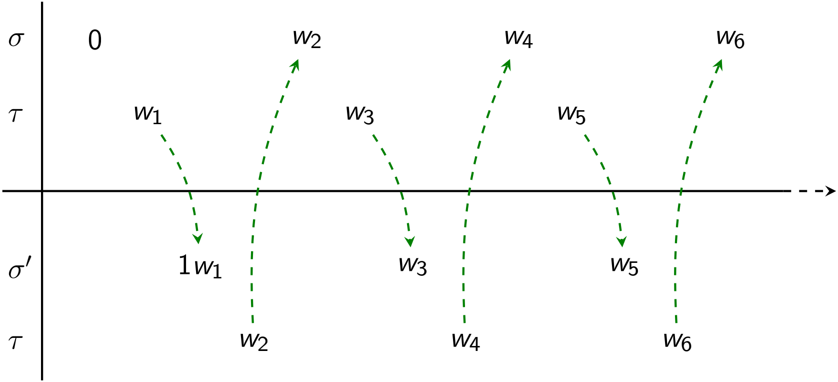

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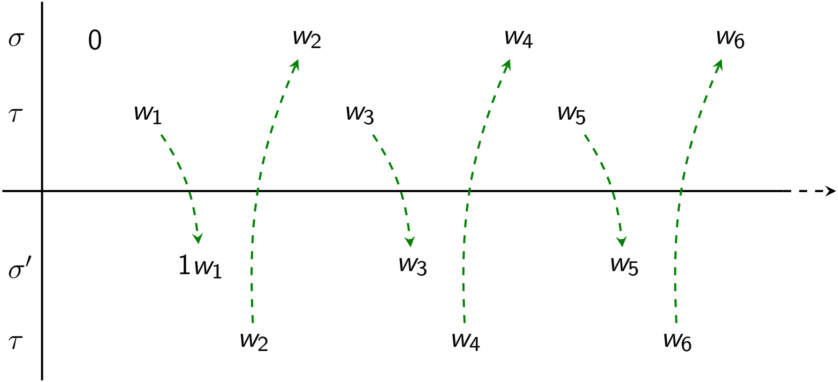

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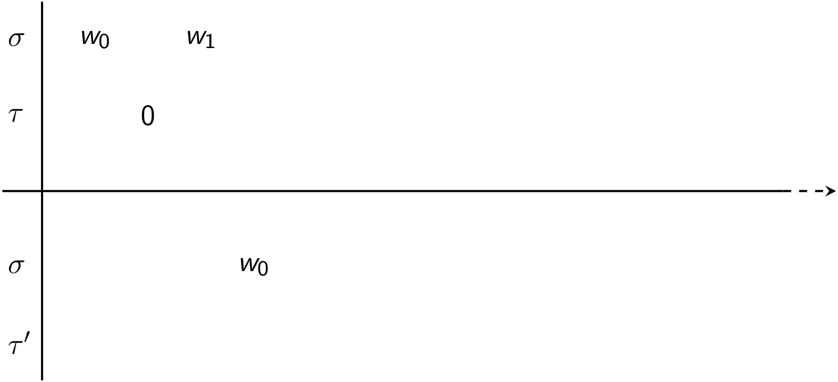

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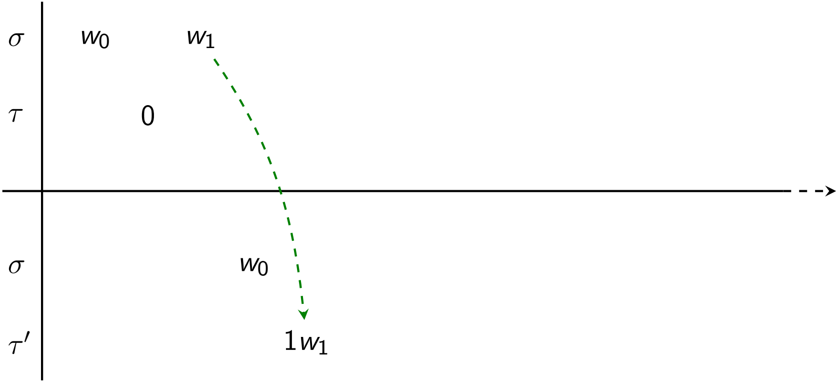

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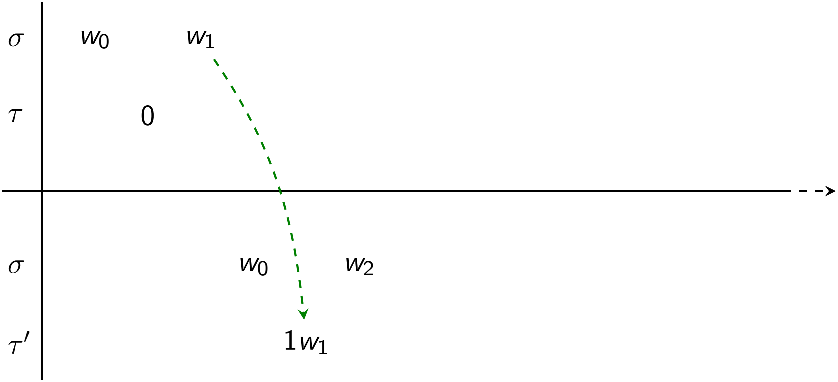

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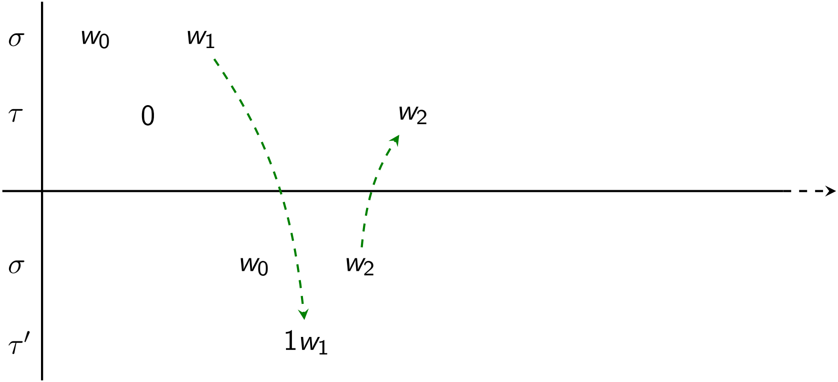

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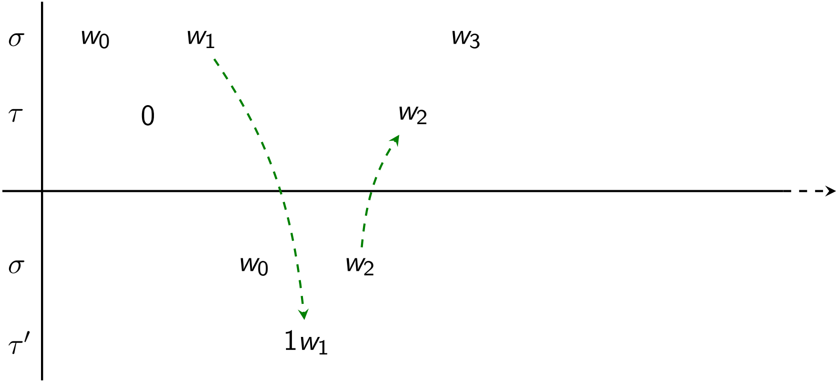

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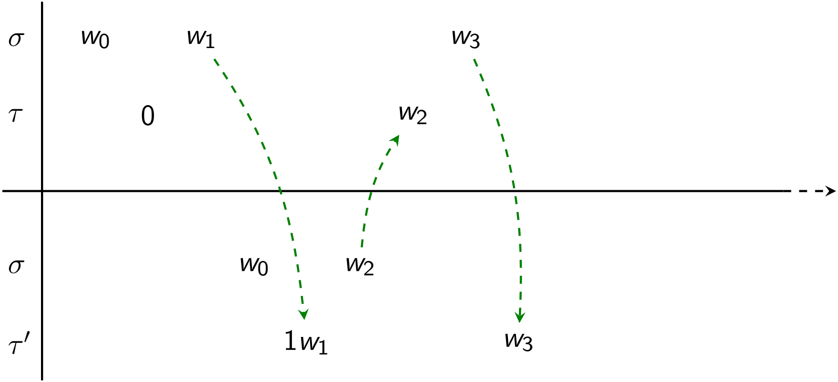

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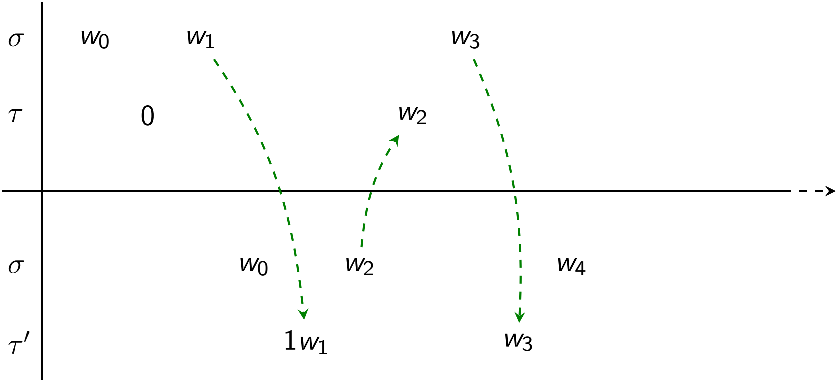

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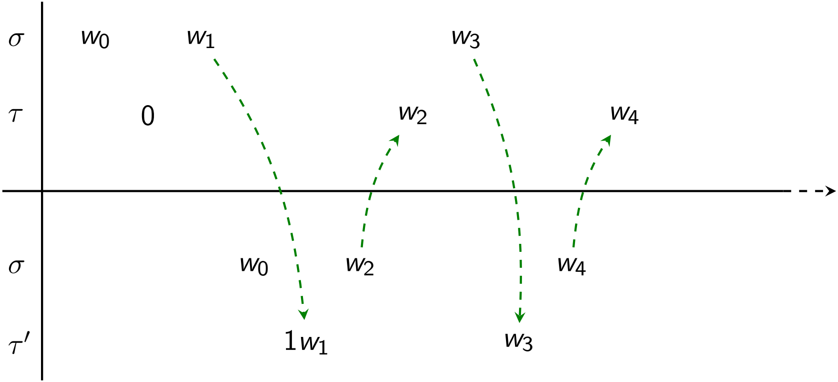

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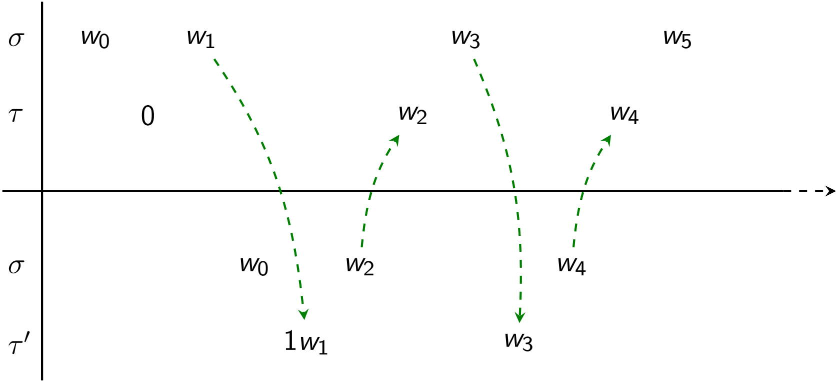

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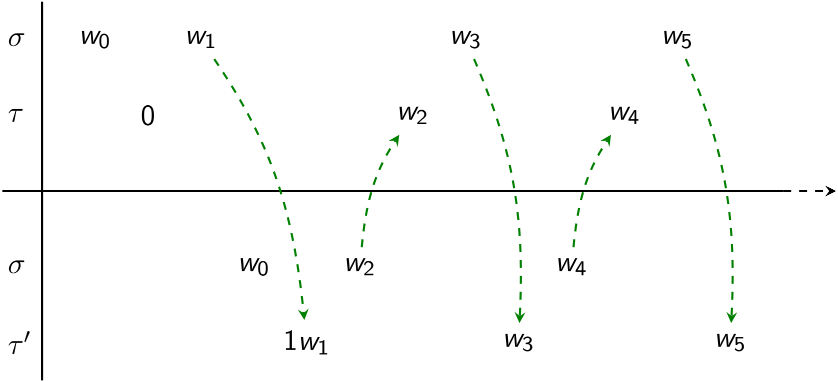

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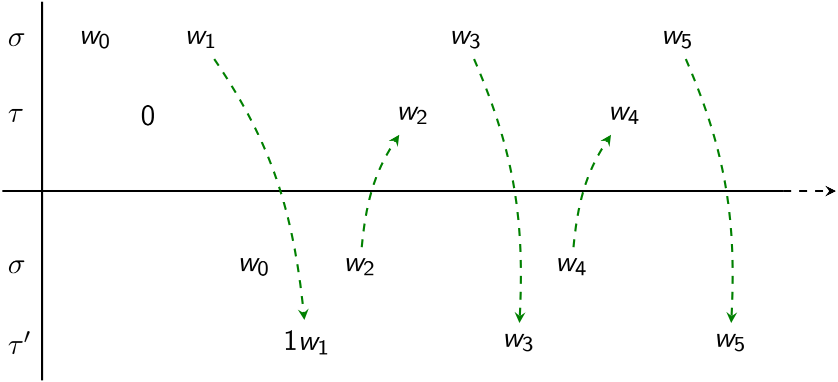

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Set Theory

- Recall that we constructed f by picking S such that it contains exactly one element of each $\sim$-equivalence class.
- This seems innocuous, but requires the *axiom of choice*.

  For every family $(S_i)_{i \in I}$ of nonempty sets there exists an indexed family $(x_i)_{i \in I}$ of elements such that $x_i \in S_i$ for every $i \in I$.

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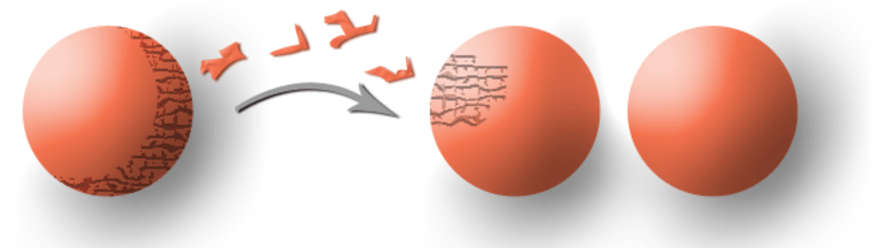
The following statements are equivalent to the axion of choice:

1. The cartesian product of a collection of non-empty sets is non-empty.
2. *Well-ordering principle*: Every set can be well-ordered.
3. *Zorn's Lemma*: Every non-empty partially ordered set in which every chain (i.e., totally ordered subset) has an upper bound contains at least one maximal element.

Consequences of the Axion of Choice

The Banach-Tarski paradox:



"Doubling of a sphere, as per the Banach-Tarski Theorem" by Sean Kelly (CC BY-SA 3.0)

You can decompose a ball into five pieces and reassemble them into two balls of the same size by just moving and rotating these pieces.

Parting Words

The Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma? *(Jerry L. Bona)*

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Merry Christmas and a happy new year