Reactive Synthesis

Lecture 2

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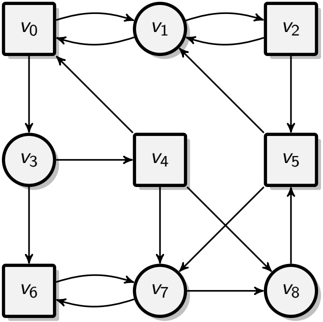
(Saarland University)

Recap

A *game* $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ consists of an arena $\mathcal{A} = (V, V_0, V_1, E)$ and a winning condition $\mathrm{Win} \subseteq V^\omega$.



$$\mathrm{Win} = \{\rho_0\rho_1\rho_2\cdots \in V^\omega \mid \exists v \in V \text{ such that } \rho_n \neq v \text{ for all } n\}$$

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- σ is positional if $\sigma(wv) = \sigma(v)$ for all $w \in V^*$ and all $v \in V_i$.
- ρ is consistent with σ if $\rho_{n+1} = \sigma(\rho_0 \cdots \rho_n)$ for every $n \in \mathbb{N}$ with $\rho_n \in V_i$.

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- σ is a winning strategy for $\mathcal{G}$ from $v \in V$ if every play that is consistent with σ and starts in v is winning for Player i.
- the winning region of Player i in $\mathcal{G}$:
  $W_i(\mathcal{G}) = \{v \in V \mid \text{Player } i \text{ has winning strategy for } \mathcal{G} \text{ from } v\}$

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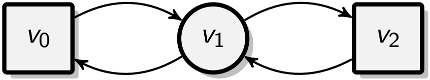
- Solving $\mathcal{G}$: determine winning regions and winning strategies.

Another Example



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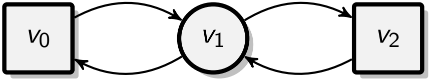

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- Player 0 has a strategy that is winning from every vertex: alternate between v_0 and v_2. Formally:

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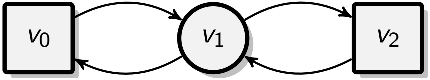

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- But both positional strategies for Player 0 are not winning from any vertex.

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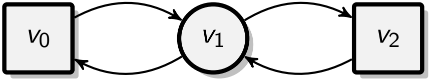

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- But both positional strategies for Player 0 are not winning from any vertex.
- Thus, positional strategies are not always sufficient.

Recap

Let $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$ be a game with vertex set V.

- $\mathcal{G}$ is a safety game if there is a set $S \subseteq V$ such that

$$\mathrm{Win} = \textsc{Safety}(S) := \{\rho \in V^\omega \mid \mathrm{Occ}(\rho) \subseteq S\}.$$

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Remark

Safety and reachability games and are dual:

	Player 0	Player 1
Safety	Avoid $V \setminus S$	Reach $V \setminus S$
Reachability	Reach R	Avoid R

Plan for Today

- Basic Games

- Algorithms & Data Structures
- Advanced Games
- Temporal Logic Synthesis

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  - Solving reachability and safety games
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Duality

Definition

Let $\mathcal{A} = (V, V_0, V_1, E)$ be an arena and let $\mathcal{G} = (\mathcal{A}, \mathrm{Win})$.

1. The *dual arena* $\overline{\mathcal{A}}$ of $\mathcal{A}$ is defined as $\overline{\mathcal{A}} = (V, V_1, V_0, E)$.
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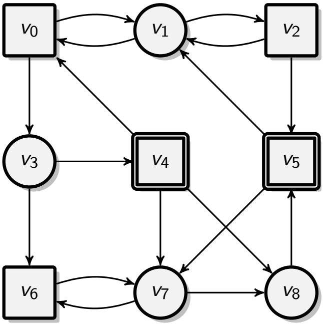
Proof.

On the blackboard. □

Solving Reachability Games



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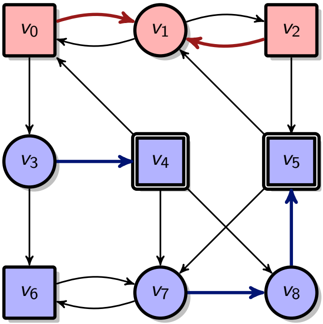

Attractor Construction

Definition

- The *controlled predecessor* $\mathrm{CPre}_0(R)$ of R is defined as

$$\begin{aligned}\mathrm{CPre}_0(R) = &\{v \in V_0 \mid v' \in R \text{ for some successor } v' \text{ of } v\} \cup \\ &\{v \in V_1 \mid v' \in R \text{ for all successors } v' \text{ of } v\}.\end{aligned}$$

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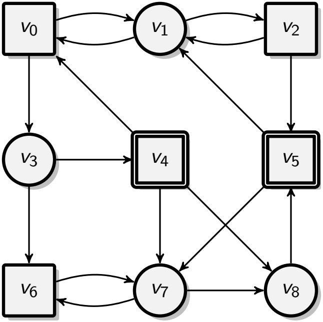
  - $\mathrm{Attr}_0(R) = \bigcup_{n \in \mathbb{N}} \mathrm{Attr}_0^n(R)$.

Example



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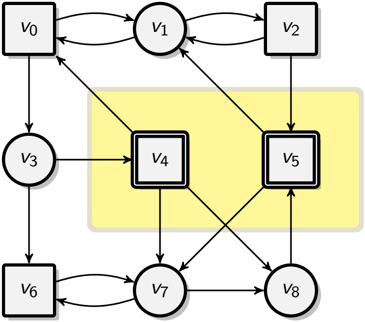

$$\mathrm{Attr}_0^0(\{v_4, v_5\}) = \{v_4, v_5\}$$

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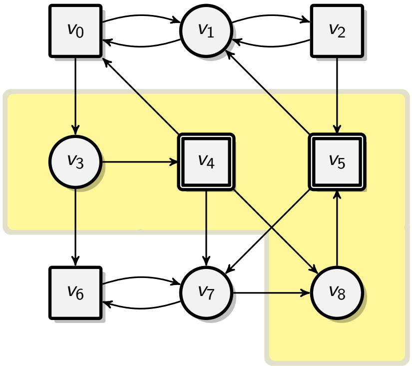
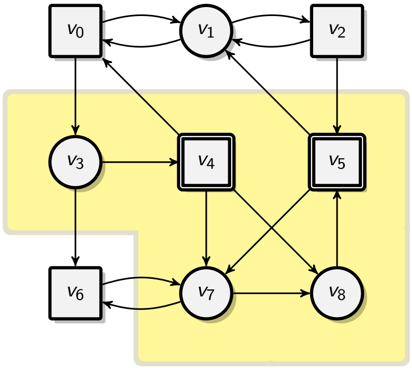


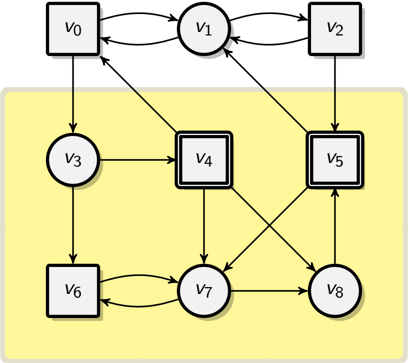
$$\mathrm{Attr}_0^1(\{v_4, v_5\}) = \{v_4, v_5\} \cup \{v_3, v_8\}$$

Example



$$\mathrm{Attr}_0^2(\{v_4, v_5\}) = \{v_3, v_4, v_5, v_8\} \cup \{v_3, v_7, v_8\}$$

Example



$$\mathrm{Attr}_0^3(\{v_4, v_5\}) = \{v_3, v_4, v_5, v_7, v_8\} \cup \{v_3, v_6, v_7, v_8\}$$

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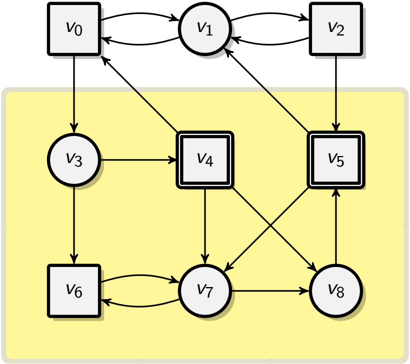

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Lemma

Let $\mathcal{G} = (\mathcal{A}, \textsc{Reach}(R))$ *be a reachability game. Then,* $W_0(\mathcal{G}) = \mathrm{Attr}_0(R)$ *and* $W_1(\mathcal{G}) = V \setminus \mathrm{Attr}_0(R)$*. In particular,* $\mathcal{G}$ *is determined.*

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Proof.

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Computing Attractors Efficiently

for all $v \in V$ **do** ▷ Initialization
    if $v \in R$ **then** $c(v) \leftarrow 0$
    else if $v \in V_0 \setminus R$ **then** $c(v) \leftarrow 1$
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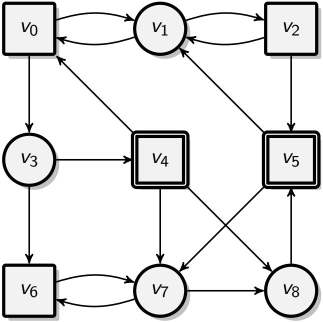
end for
$A = R$
while $A \neq \emptyset$ **do** ▷ Main Loop
    remove some v from A
    for all $(v', v) \in E$ with $c(v') > 0$ **do**
        $c(v') \leftarrow c(v') - 1$
        if $c(v') = 0$ **then** $A \leftarrow A \cup \{v'\}$
        end if
    end for
end while
return $\{v \mid c(v) = 0\}$

An Example



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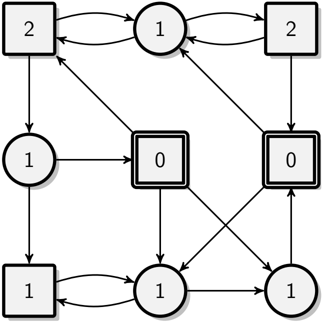

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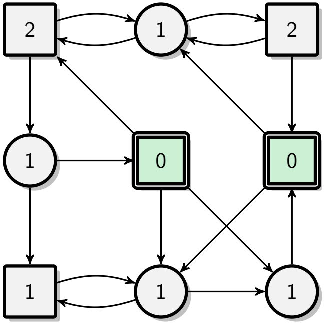

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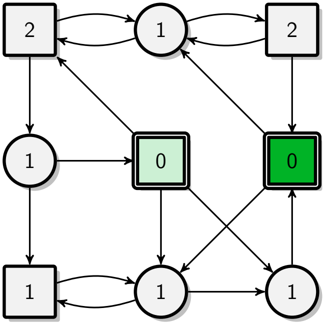

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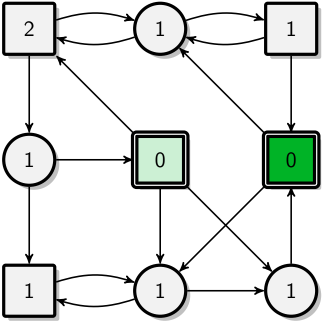

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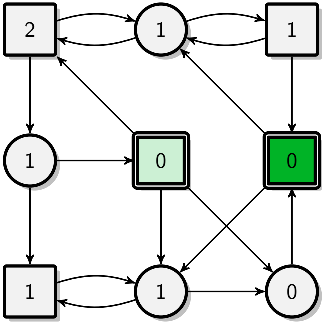

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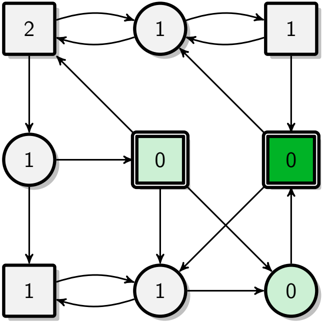

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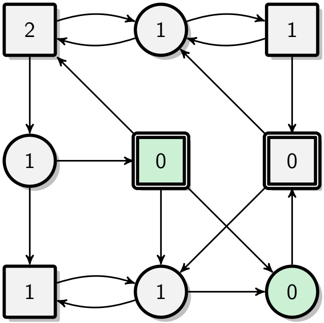

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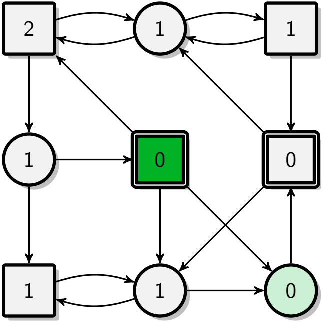

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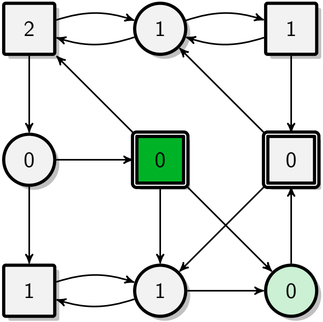

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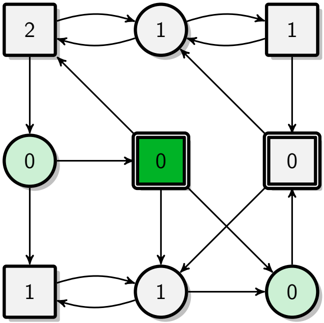

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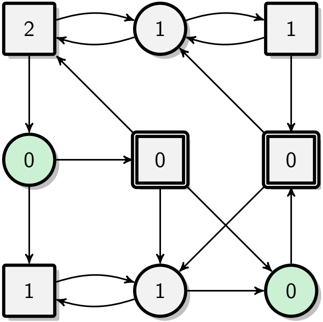

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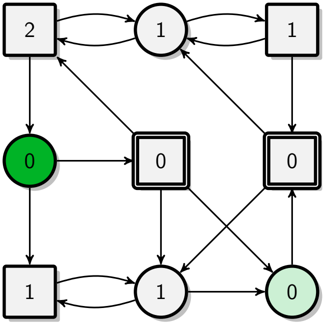

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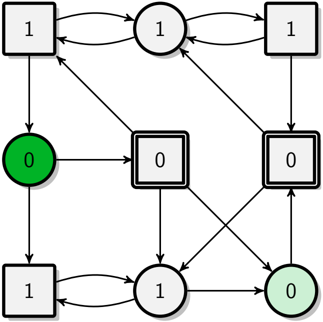

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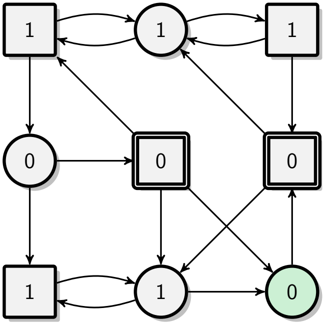

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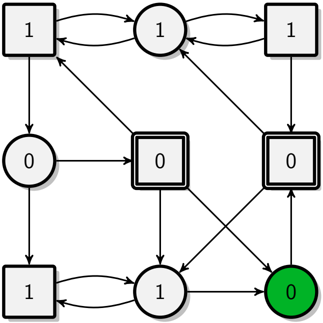

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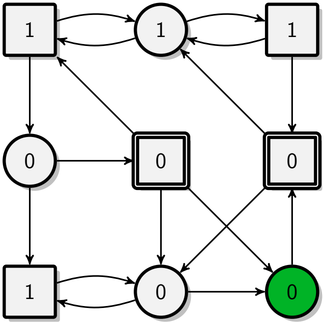

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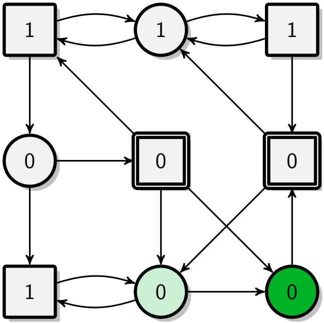

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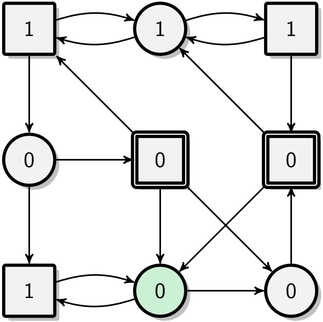

- *A* in pale green
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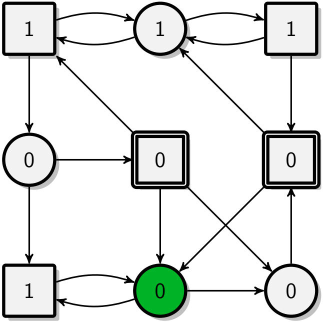

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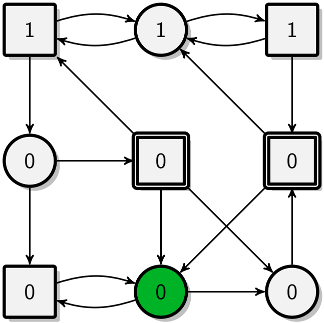

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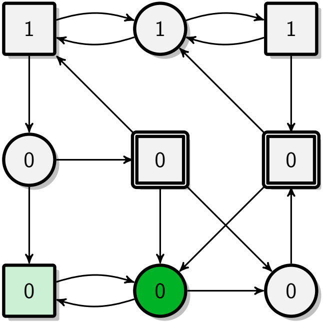

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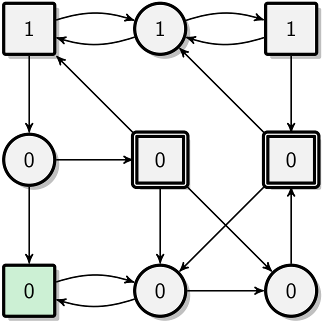

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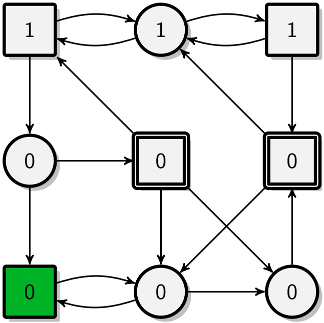

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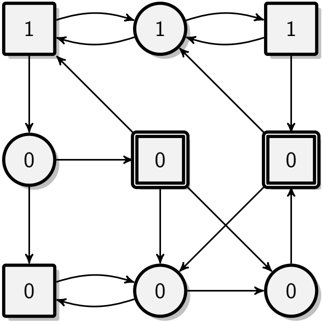

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Sketch: By constructing winning strategies, show that..

- .. the output is a subset of $W_0(\mathcal{G})$.
- .. the complement of the output is a subset of $W_1(\mathcal{G})$.

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Hence, the overall running time is bounded by $|E|$.

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Remark

The algorithm can easily be modified to compute winning strategies for both players, without increasing the running time.

- For Player 0, pick successor that caused decrease of $c(v)$.
- For Player 1, pick successor v' with $c(v') > 0$.

Announcements

Exam dates

- Exam: January 31st, 2018
- Re-exam: March 20th, 2018