Reactive Synthesis

Lecture 11

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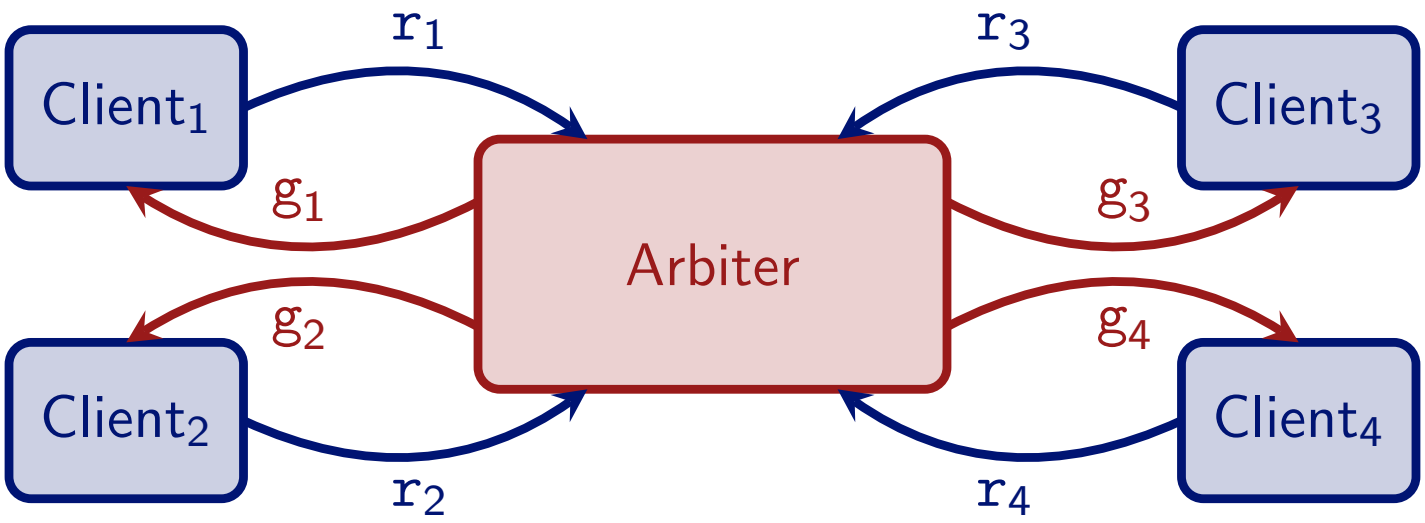
Plan for Today

- Basic Games
- Algorithms & Data Structures
- Advanced Games
- Temporal Logic Synthesis

Plan for Today

- Basic Games
- Algorithms & Data Structures
- Advanced Games
- Temporal Logic Synthesis
  - Solving the LTL Synthesis Problem:
    - Bounded Synthesis
    - Reducing Bounded Synthesis to SMT/QBF/SAT Solving

Recap: Motivating Example



Typical requirements on an arbiter:

- Every request of a client is eventually granted by the arbiter.
- Never two grants at the same time.
- No spurious grants, i.e., arbiter only gives grant to client i if it has an open request.
- A client may only pose a request if it has no open request.

Recap: Arbiter Specification in LTL

Example specifications

1. Answer every request: $\bigwedge_i \mathbf{G}(\mathtt{r}_i \rightarrow \mathbf{F}\, \mathtt{g}_i)$
2. At most one grant at a time: $\mathbf{G} \bigwedge_{i \neq j} \neg(\mathtt{g}_i \wedge \mathtt{g}_j)$
3. No spurious grants:

$$\bigwedge_i \neg[\,(\neg \mathtt{r}_i \mathbf{U} (\neg \mathtt{r}_i \wedge \mathtt{g}_i))\,] \wedge \neg[\,\mathbf{F}(\mathtt{g}_i \wedge \mathbf{X}(\neg \mathtt{r}_i \mathbf{U} (\neg \mathtt{r}_i \wedge \mathtt{g}_i)))\,]$$

4. No new requests while request is open:
   $\bigwedge_i \mathbf{G}(\mathtt{r}_i \rightarrow \mathbf{X}(\neg \mathtt{r}_i \,\mathbf{U}\, \mathtt{g}_i))$

Recap: LTL Games

Notation

Given a labeling $\ell\colon V \to 2^P$ of V by atomic propositions and a play $\rho = \rho_0\rho_1\rho_2\cdots$ define $\ell(\rho) = \ell(\rho_0)\ell(\rho_1)\ell(\rho_2)\cdots$.

Definition

Let $\mathcal{A} = (V, V_0, V_1, E)$ be an arena and let $\ell\colon V \to 2^P$ be a labeling of $\mathcal{A}$'s vertices by atomic propositions. Then, the LTL condition $\mathrm{LTL}(\varphi, \ell)$ is defined as

$$\mathrm{LTL}(\varphi, \ell) := \{\rho \in V^\omega \mid (\ell(\rho), 0) \models \varphi\}.$$

We call a game $\mathcal{G} = (\mathcal{A}, \mathrm{LTL}(\varphi, \ell))$ an *LTL game*.

Recap: Main Result

Theorem

LTL games are determined with finite-state strategies of doubly-exponential size (in the size of the formula) and can be solved in doubly-exponential time.

Remark

Both lower bounds are tight.

Recap: Main Result

Theorem

LTL games are determined with finite-state strategies of doubly-exponential size (in the size of the formula) and can be solved in doubly-exponential time.

Remark

Both lower bounds are tight.

Problem: size of game doubly exponential in specification, even if solution is very simple. (and simple solutions will be hard to find)

Bounded Synthesis

Synthesis:
Is there an implementation that satisfies the specification?

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Is there an implementation *with at most N states* that satisfies the specification?

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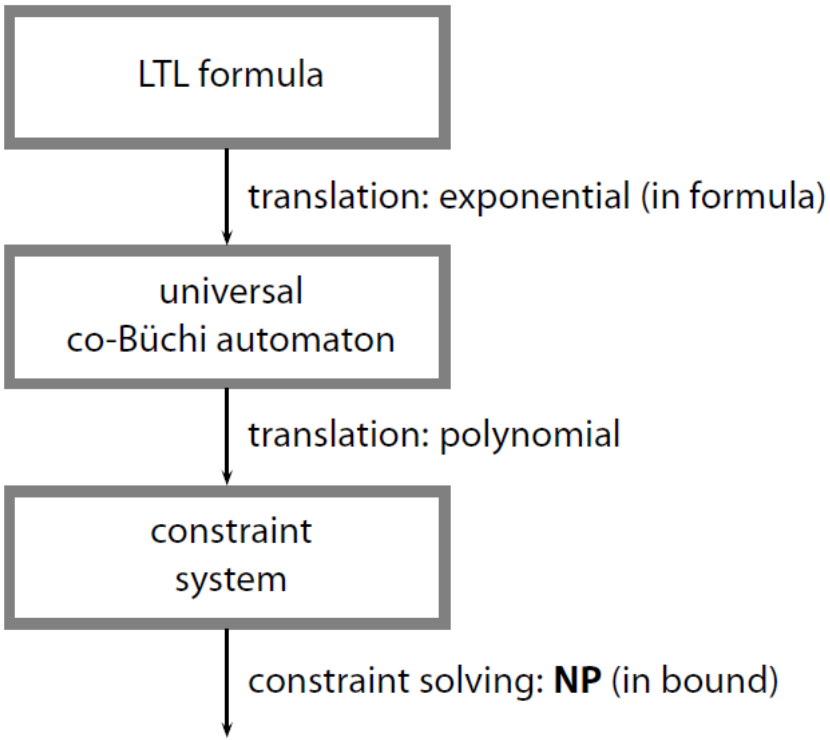
Bounded Synthesis:
Is there an implementation *with at most N states* that satisfies the specification?

Idea: Search for **simple** implementations first, find them **fast**.

Bounded LTL Synthesis



Input Complexity:
Exponential in the size of the specification.

Output Complexity:
NP-complete in the size of the implementation.

Labeled Transition Systems

Let I be a finite set of boolean input variables and O a finite set of boolean output variables.

Definition

A **labeled transition system** is a tuple $\mathcal{T} = (T, t_0, \tau, o)$, where

- T is a set of states,
- $t_0 \in T$ is the initial state,
- $\tau : T \times \mathbb{B}^I \to T$ is the transition function, and
- $o : T \to \mathbb{B}^O$ is the labeling function.

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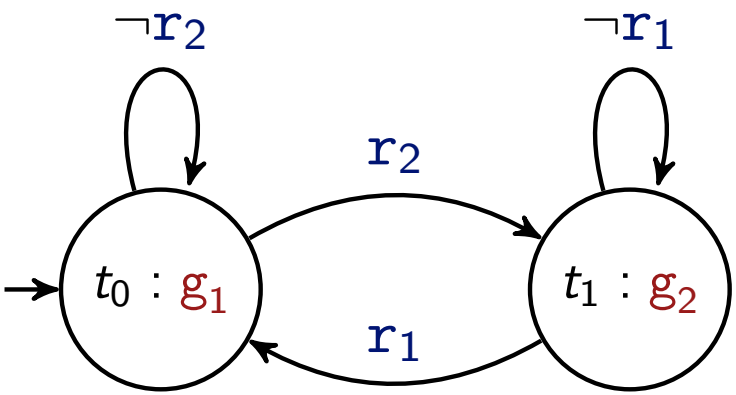
Connection to games:

Labeled transition systems correspond to finite-state strategies: choice of next state in game corresponds to choice of output values of transition system.
(like for safety games, but with an underlying memory structure)

Example: Labeled Transition System

Input variables: $I = \{\mathtt{r}_1, \mathtt{r}_2\}$
Output variables: $O = \{\mathtt{g}_1, \mathtt{g}_2\}$
States: $T = \{t_0, t_1\}$
Labeling: $o(t_0) = \{\mathtt{g}_1, \neg\mathtt{g}_2\}, o(t_1) = \{\neg\mathtt{g}_1, \mathtt{g}_2\}$



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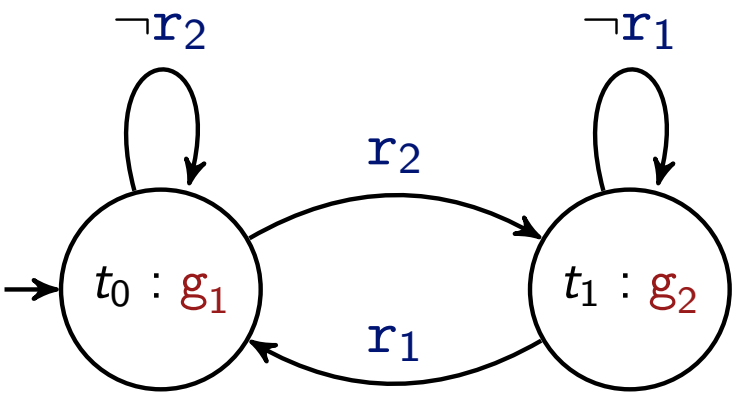

Example run

Env:

Sys:

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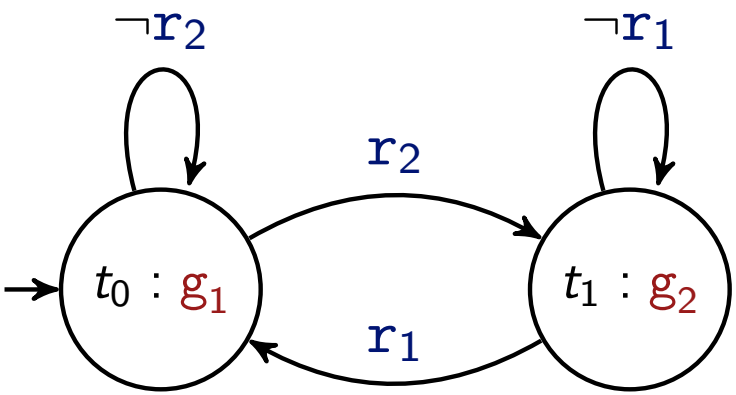

Example run

Env:
Sys: $\mathtt{g}_1$

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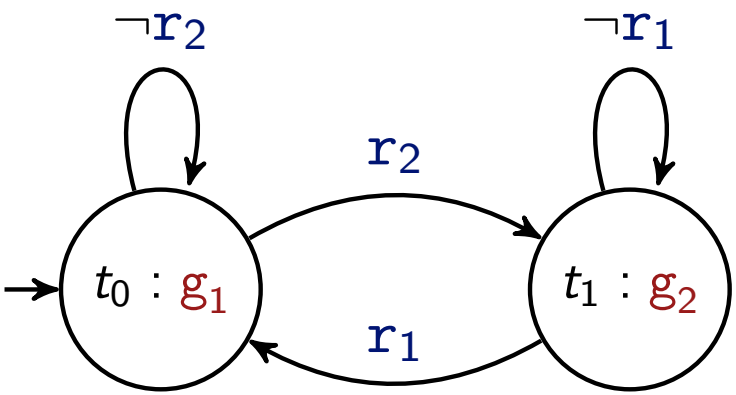

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Env: $\mathtt{r}_1, \mathtt{r}_2$
Sys: $\mathtt{g}_1$

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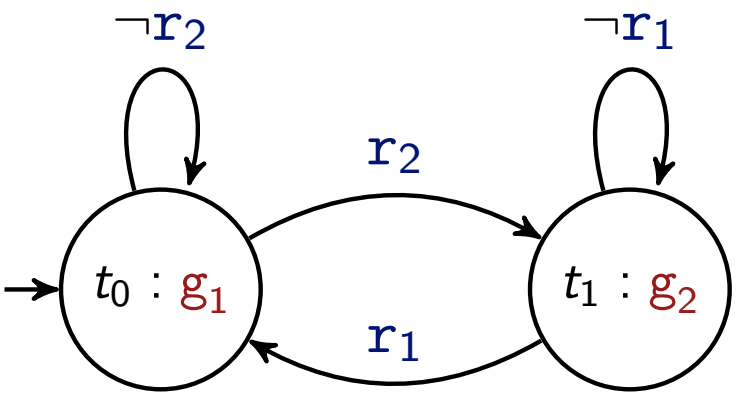

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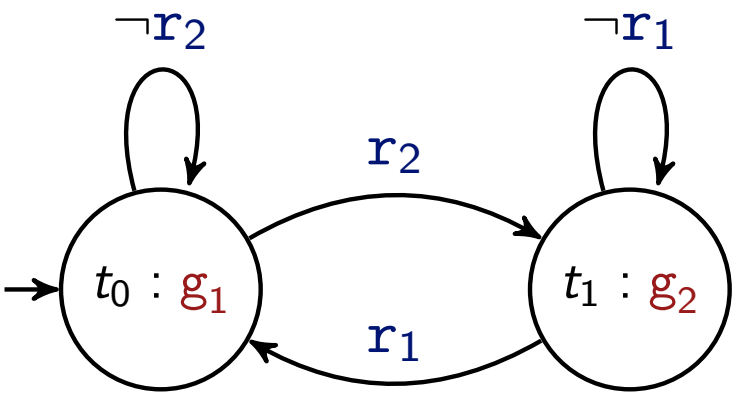

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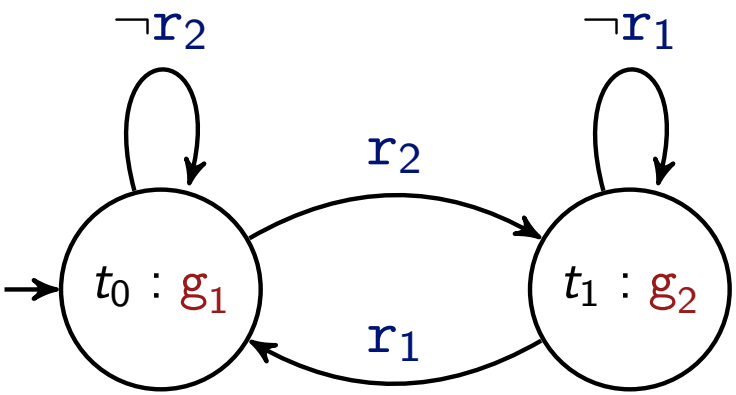

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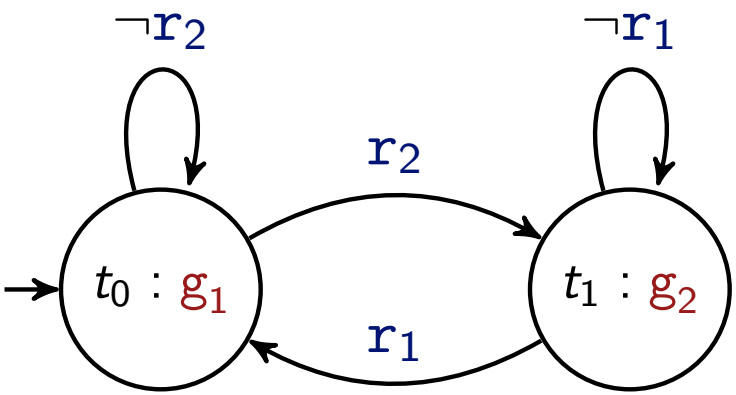

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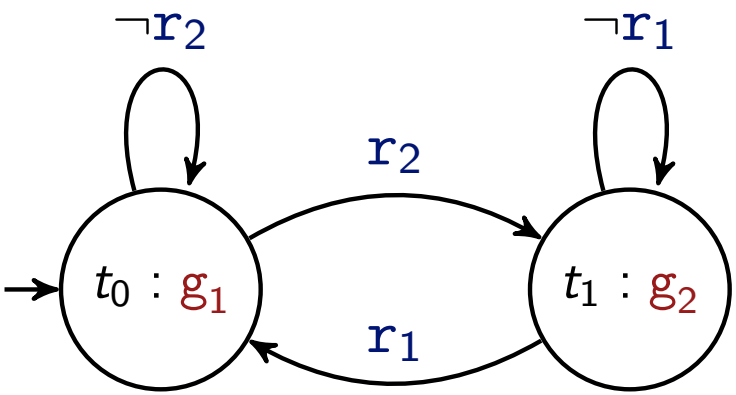

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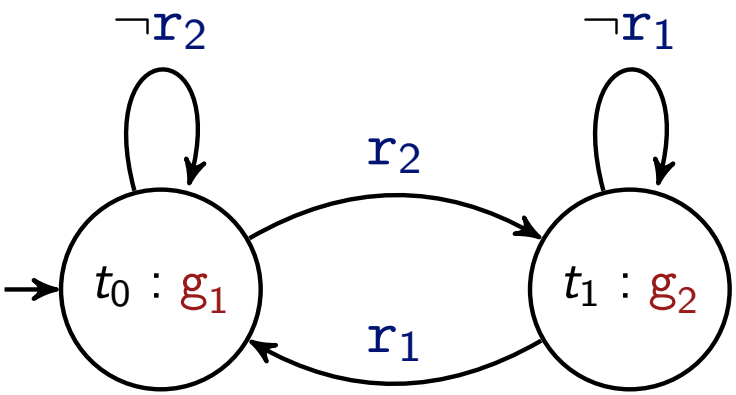

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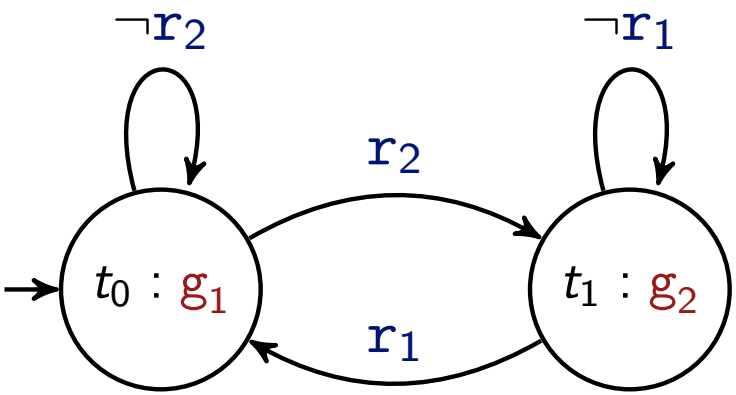

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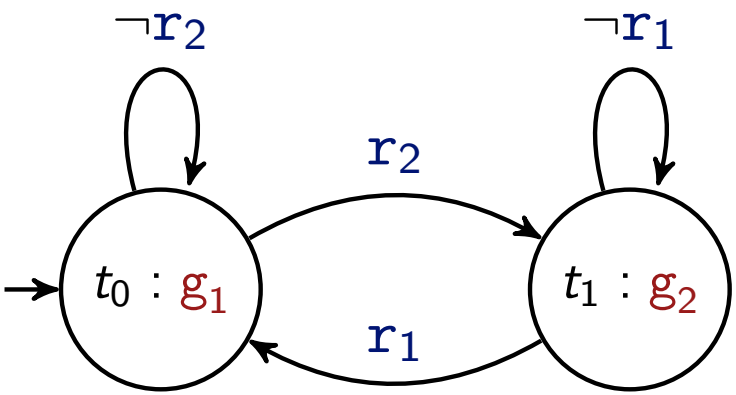

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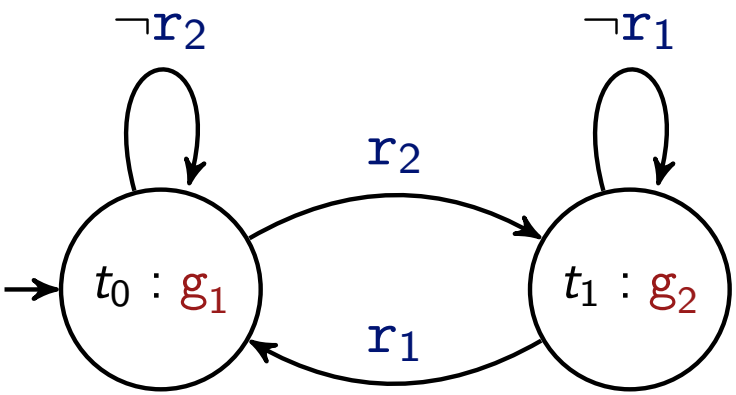

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Env:	$\mathtt{r}_1, \mathtt{r}_2$		$\mathtt{r}_1$	$\mathtt{r}_1$	–	–
Sys:		$\mathtt{g}_1$	$\mathtt{g}_2$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$

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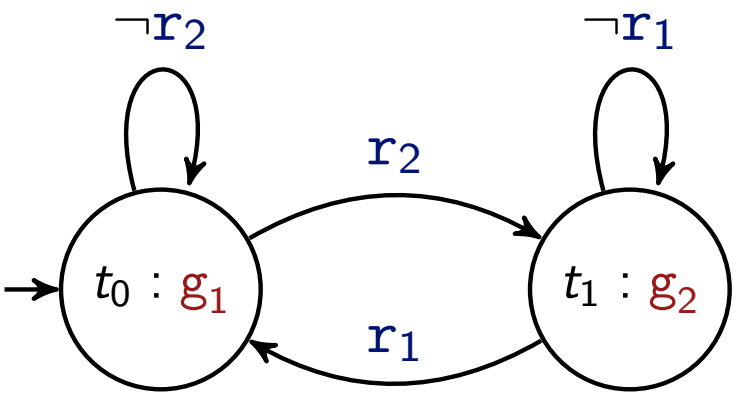

Example run

Env:	$\mathtt{r}_1, \mathtt{r}_2$	$\mathtt{r}_1$	$\mathtt{r}_1$	–	–	$\mathtt{r}_2$
Sys:	$\mathtt{g}_1$	$\mathtt{g}_2$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$

Example: Labeled Transition System

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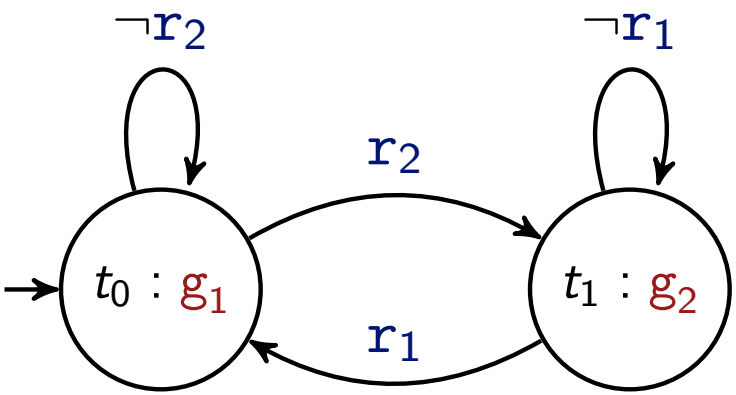

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Env:	$\mathtt{r}_1, \mathtt{r}_2$		$\mathtt{r}_1$	$\mathtt{r}_1$	–	–	$\mathtt{r}_2$	
Sys:		$\mathtt{g}_1$	$\mathtt{g}_2$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_2$

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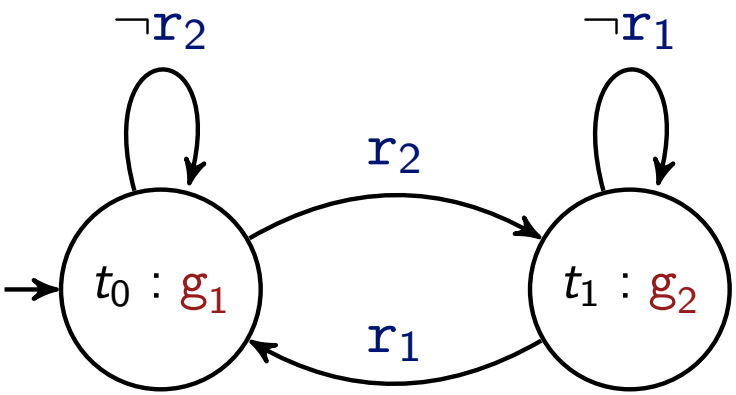

Example run

Env:	r_1, r_2	r_1	r_1	–	–	r_2	–
Sys:	g_1	g_2	g_1	g_1	g_1	g_1	g_2

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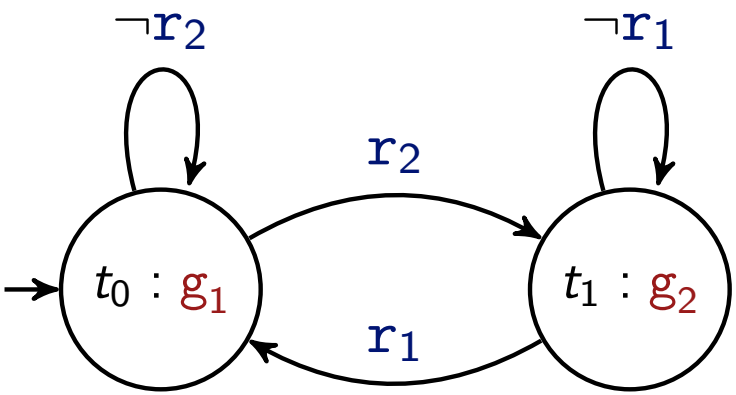

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Sys:		$\mathtt{g}_1$	$\mathtt{g}_2$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_2$	$\mathtt{g}_2$

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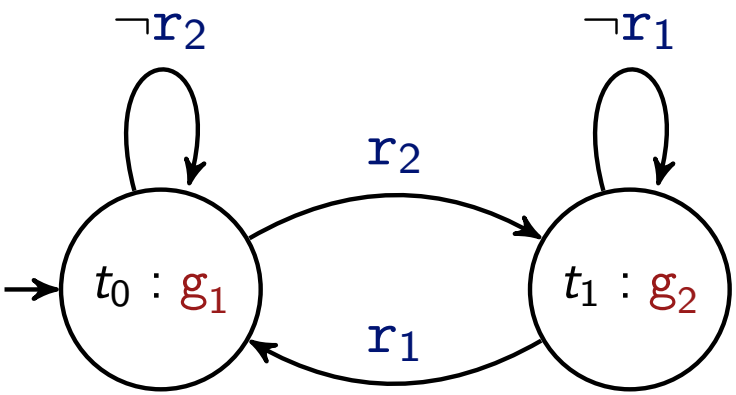

Example run

Env:	$\mathtt{r}_1, \mathtt{r}_2$	$\mathtt{r}_1$	$\mathtt{r}_1$	$-$	$-$	$\mathtt{r}_2$	$-$	$\mathtt{r}_1, \mathtt{r}_2$
Sys:	$\mathtt{g}_1$	$\mathtt{g}_2$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_2$	$\mathtt{g}_2$

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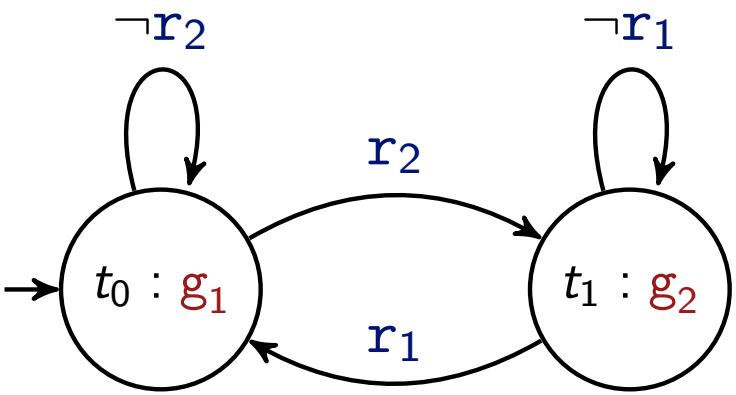

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Env:	$\mathtt{r}_1, \mathtt{r}_2$	$\mathtt{r}_1$	$\mathtt{r}_1$	$-$	$-$	$\mathtt{r}_2$	$-$	$\mathtt{r}_1, \mathtt{r}_2$	
Sys:	$\mathtt{g}_1$	$\mathtt{g}_2$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_2$	$\mathtt{g}_2$	$\mathtt{g}_1$

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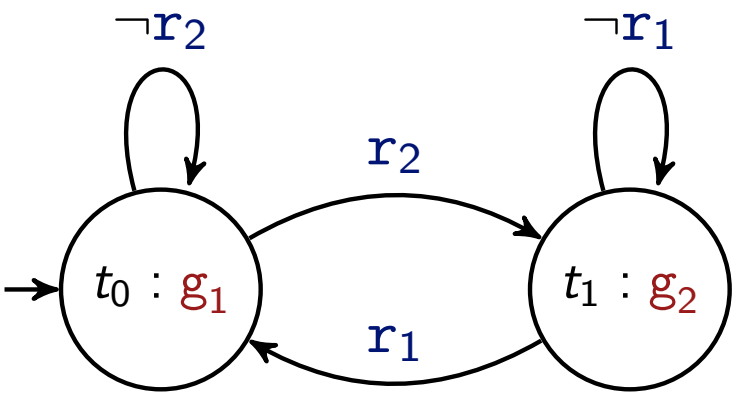

Example run

Env:	$\mathtt{r}_1, \mathtt{r}_2$	$\mathtt{r}_1$	$\mathtt{r}_1$	–	–	$\mathtt{r}_2$	–	$\mathtt{r}_1, \mathtt{r}_2$	–
Sys:	$\mathtt{g}_1$	$\mathtt{g}_2$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_1$	$\mathtt{g}_2$	$\mathtt{g}_2$	$\mathtt{g}_1$

Universal Co-Büchi Automata

Let I and O be finite sets of boolean variables. We will run the UCB on an LTS, so O are inputs and I are outputs for the UCB.

Definition

A **universal co-Büchi automaton (UCB)** is a tuple $\mathcal{U} = (Q, q_0, \delta, F)$, where

- Q is a finite set of states,
- $q_0 \in Q$ is the initial state,
- $\delta : Q \times \mathbb{B}^O \to \mathbb{B}^{(Q \times \mathbb{B}^I)}$ is the transition function (that maps a state and an input to a set of tuples of successor states and outputs), and
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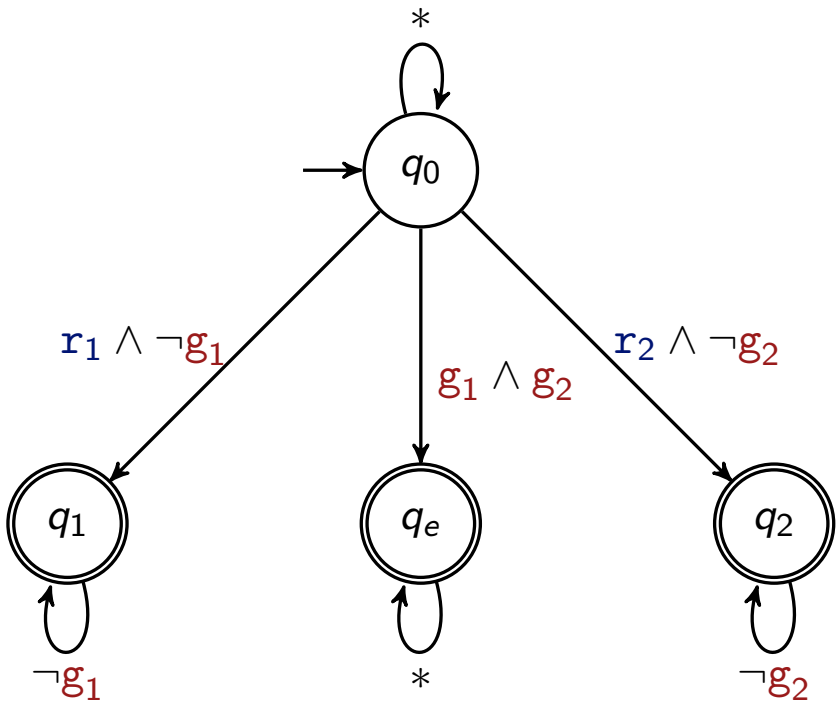
Example: Universal Co-Büchi Automaton

UCB for basic arbiter specification (with two clients):

$\bigwedge_{i \in \{1,2\}} \mathbf{G}(\mathtt{r}_i \rightarrow \mathbf{F}\, \mathtt{g}_i)$
$\mathbf{G}\, \neg(\mathtt{g}_1 \wedge \mathtt{g}_2)$

Input variables $O = \{\mathtt{g}_1, \mathtt{g}_2\}$
Output variables $I = \{\mathtt{r}_1, \mathtt{r}_2\}$



From LTL to UCB

To obtain a UCB that accepts those LTSs that satisfy a specification φ, do the following:

1. Construct the non-deterministic Büchi automaton $\mathcal{B} = (Q, \Sigma, q_0, \Delta, F)$ for $\neg\varphi$, with $\Sigma = \mathbb{B}^{I \cup O}$.
2. The desired UCB is $\mathcal{U} = (Q, q_0, \delta, F)$, where δ is defined by: $(q', i) \in \delta(q, o)$ if and only if $(q, \sigma, q') \in \Delta$ and $\sigma \cap I = i$ and $\sigma \cap O = o$.

Run Graph, Acceptance of an LTS

Definition

A **run graph** of a UCB $\mathcal{U} = (Q, q_0, \delta, F)$ on a LTS $\mathcal{T} = (T, t_0, \tau, o)$ is the directed graph $G = (V, E)$ that satisfies:

- $V \subseteq Q \times T$,
- $(q_0, t_0) \in V$, and
- if $(q, t) \in V$, then $((q, t), (q', \tau(t, i))) \in E$ if and only if $(q', i) \in \delta(q, o(t))$.

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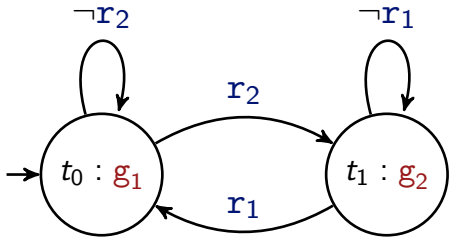
$\mathcal{T}$ is **accepted** by $\mathcal{U}$ if every infinite path in the run graph visits states in F only finitely often.

Example: Run Graph

LTS:



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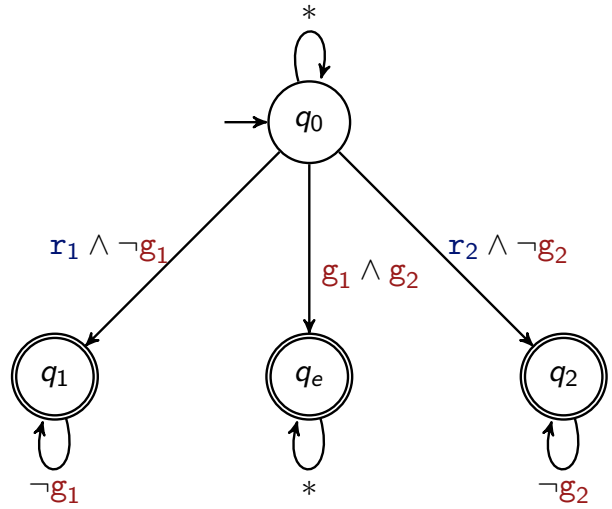

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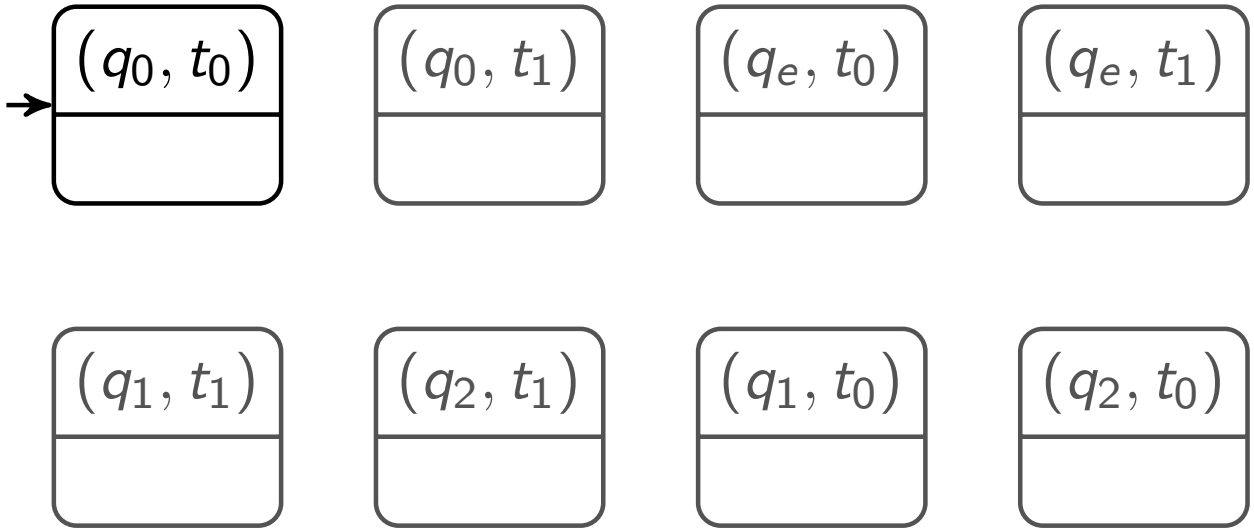

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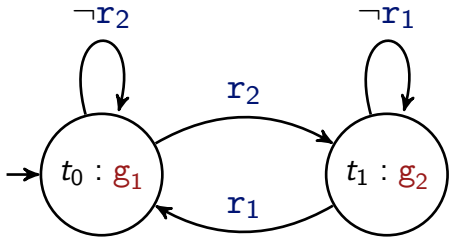

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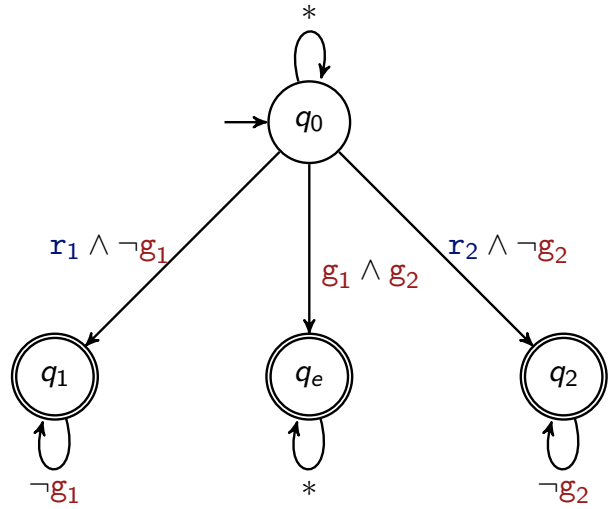

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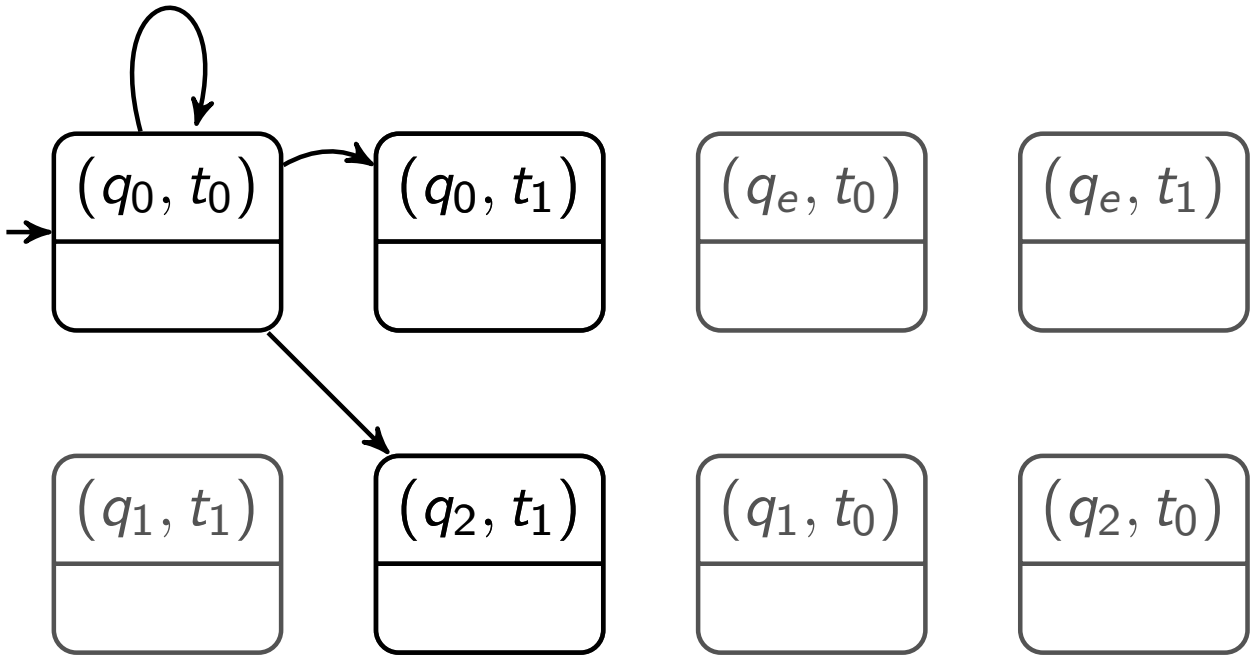

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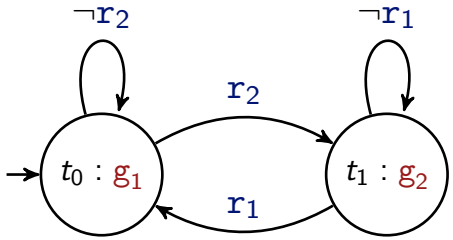

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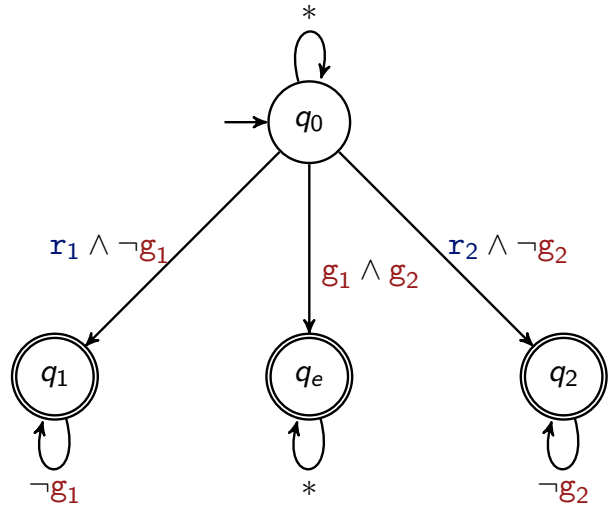

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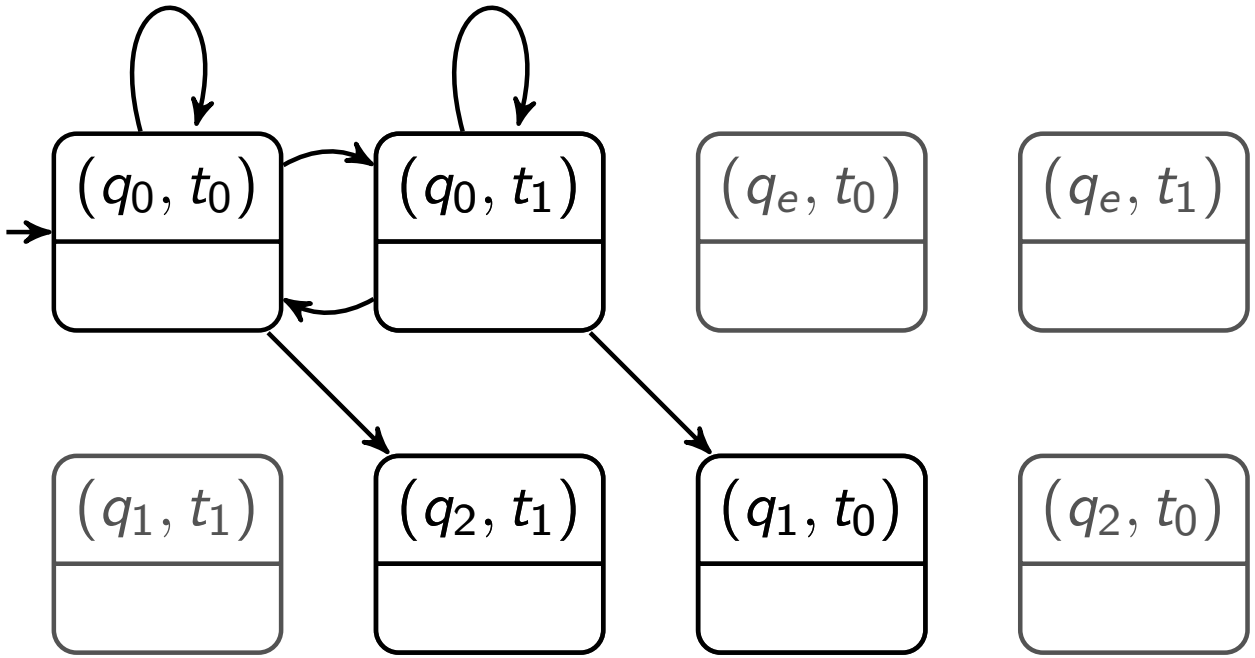

Valid Annotations for Run Graphs

Definition

For given UCB $\mathcal{U} = (Q, q_0, \delta, F)$ and LTS $\mathcal{T} = (T, t_0, \tau, o)$, an **annotation** is a function $\lambda : Q \times T \to \{\bot\} \cup \mathbb{N}$.

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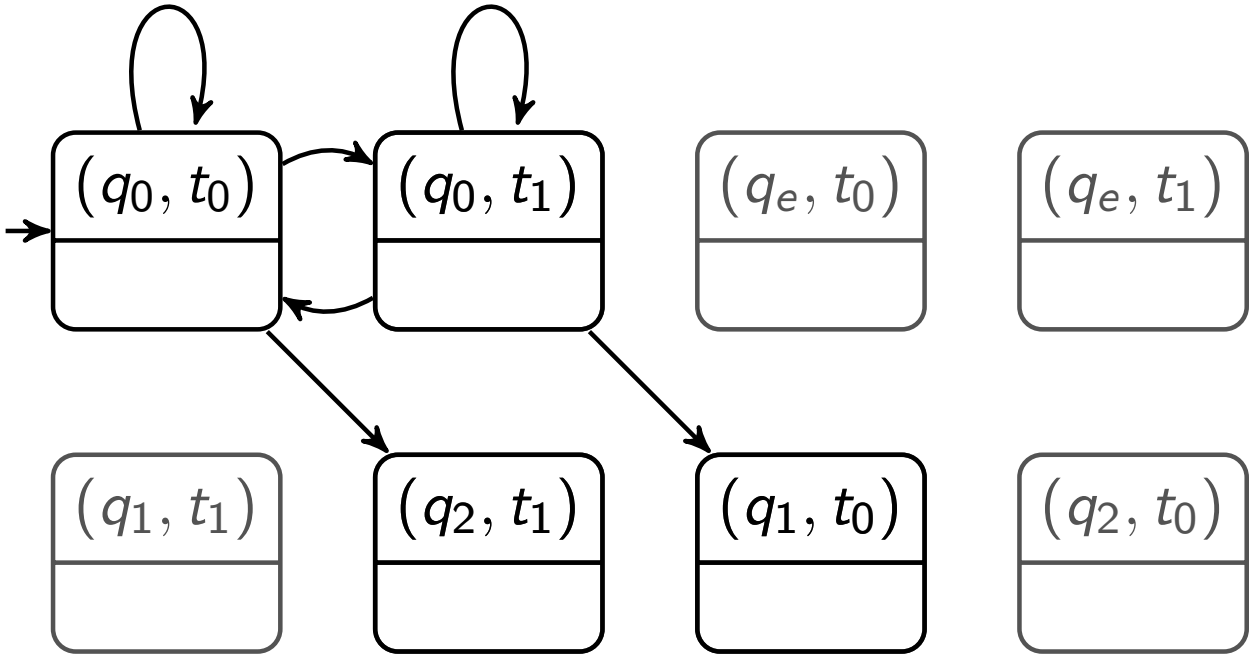
An annotation is **c-bounded** if $\lambda(q, t) \leq c$ for all q, t (assuming $\bot < c$). An annotation is **bounded** if it is c-bounded for some c.

Example: Annotated Run Graph

Run graph with valid annotation λ:



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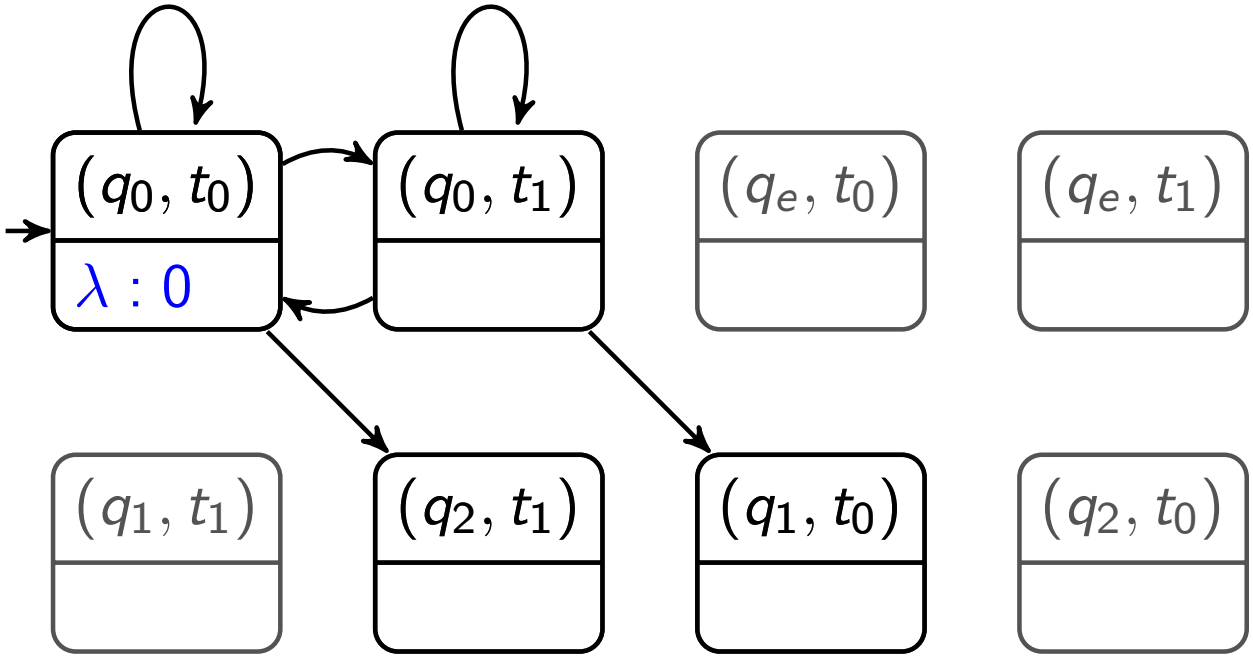

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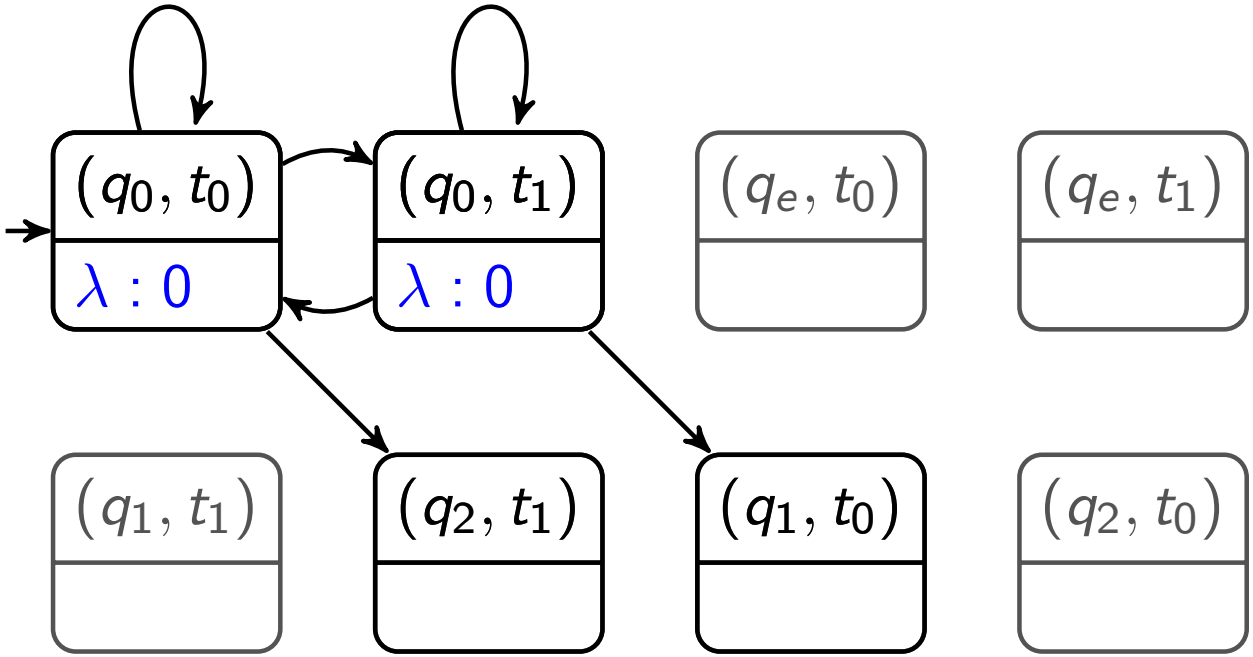

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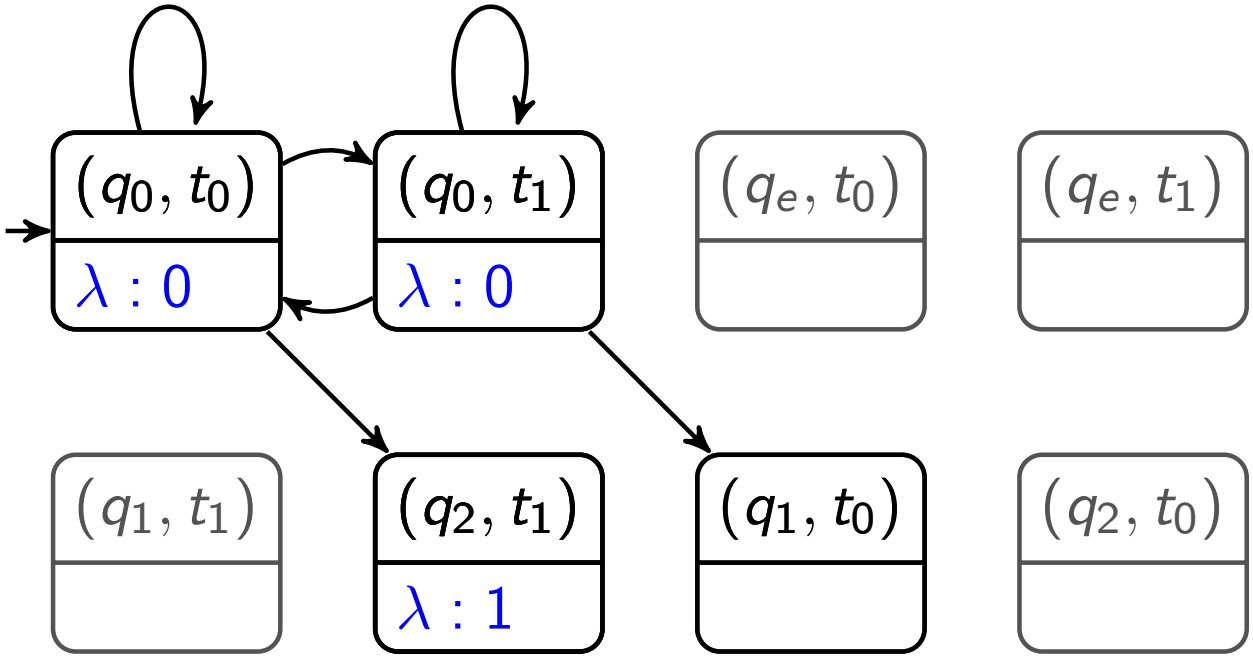

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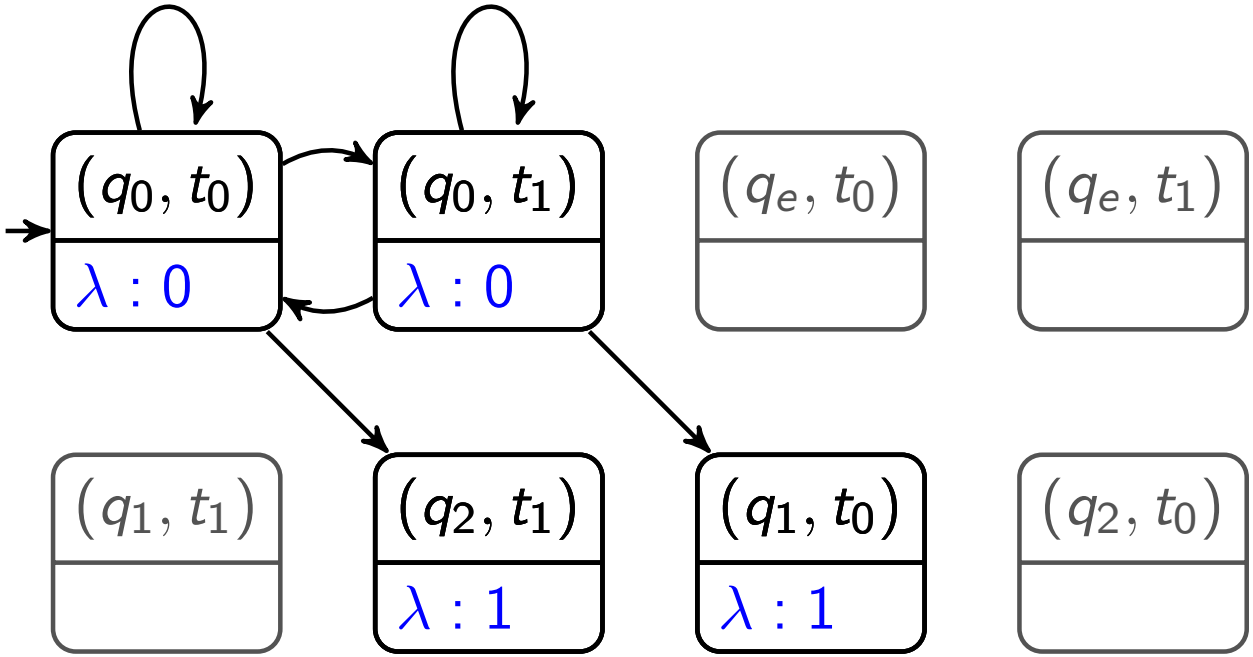

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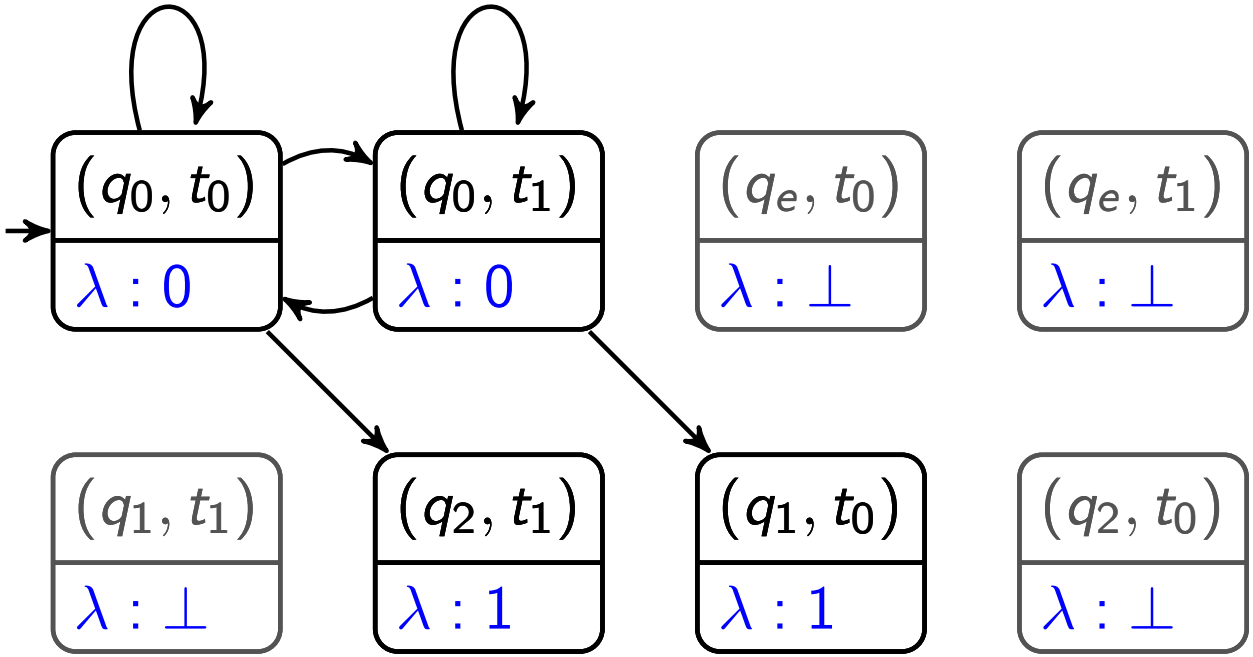

Bounded Valid Annotations and Acceptance

Theorem

A finite-state LTS $\mathcal{T} = (T, t_0, \tau, o)$ *is accepted by a UCB* $\mathcal{U} = (Q, q_0, \delta, F)$ *if and only if it has a valid* $(|T| \cdot |F|)$*-bounded annotation.*

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- if G has a lasso with a rejecting state in its loop, then there cannot be a valid bounded annotation (since at some point the value would have to be strictly smaller than itself)
- if G does not have such a lasso, then assign to each $(q, t) \in G$ the highest number of rejecting states that occur on some path from (q_0, t_0), and assign $\bot$ to unreachable vertices. The assigned values will be smaller than $(|T| \cdot |F|)$ (otherwise we would have a loop).

Existence of Bounded Valid Annotations

Definition

An LTS $\mathcal{T} = (T, t_0, \tau, o)$ with output variables O and input variables $I \subseteq O$ is **input-preserving** if for all $t \in T$ and inputs $i \in \mathbb{B}^I$: $o(\tau(t, i)) \cap I = i$.

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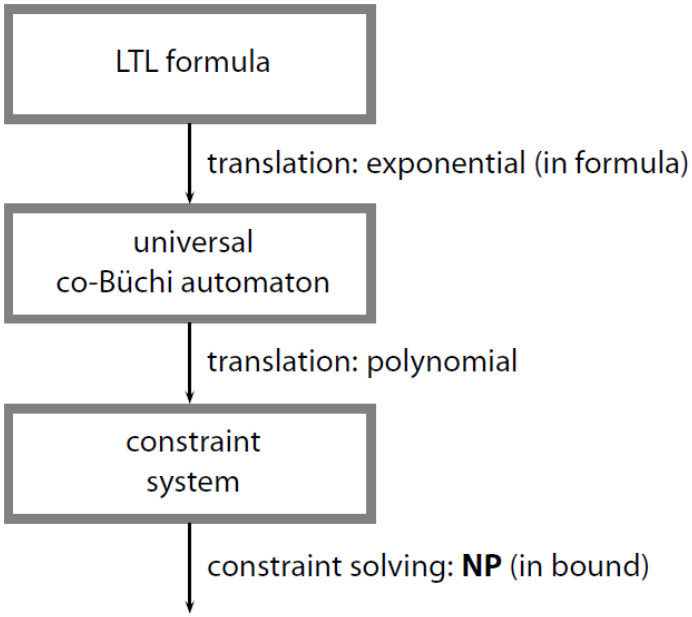
How to find such an accepted LTS?

Synthesis of Systems with Bounded Size



Idea: Fix bound c, search for LTS with at most c states and a c-bounded valid annotation.

Approach: Define **constraint system** that specifies existence of LTS and annotation, use a **constraint solver** to find a solution.

Encoding Bounded Synthesis

To encode the transition system, introduce:

- a transition function $\tau : T \times 2^I \to T$
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These functions are unknown - we will define a constraint system that defines the kind of functions we are looking for, and ask a constraint solver to find a solution.

Constraint System for Bounded Synthesis

$$\lambda^{\mathbb{B}}_{q_0}(t_0) \wedge \lambda^{\#}_{q_0}(t_0) = 0$$

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$\forall a \in I, \forall i \in \mathbb{B}^I, \forall t \in \{0, \ldots, c\}$:

$$a(\tau(t, i)) \leftrightarrow a \in i$$

Correctness of Constraint System

Theorem

The constraint system is satisfiable (modulo a theory with order) if and only if there exists an input-preserving finite-state transition system that is accepted by $\mathcal{U}$.

Moreover, a satisfying assignment to the functions τ and $\vec{a}$ represents such an LTS.

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How are such constraint systems solved?

SMT Solving

Constraint systems generated by Bounded Synthesis can be solved by SMT (Satisfiability modulo Theories) solvers.

Theories can be equality, uninterpreted functions, (linear/nonlinear, real/integer) arithmetic, etc.

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Basic approach:

- abstract the problem by interpreting all atomic statements as independent boolean variables
- use SAT solving to find a satisfying assignment
- check if satisfying assignment holds in given theory:
  - if it does, return the solution
  - if not, add a theory lemma (excluding this solution at abstract level) and restart

Alternatives: QBF and SAT encoding

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- SAT (basic): introduce a boolean variable for every possible **application** of a function symbol to an argument. Universal quantifiers need to be unrolled. Satisfying assignments can be lifted to functions again.

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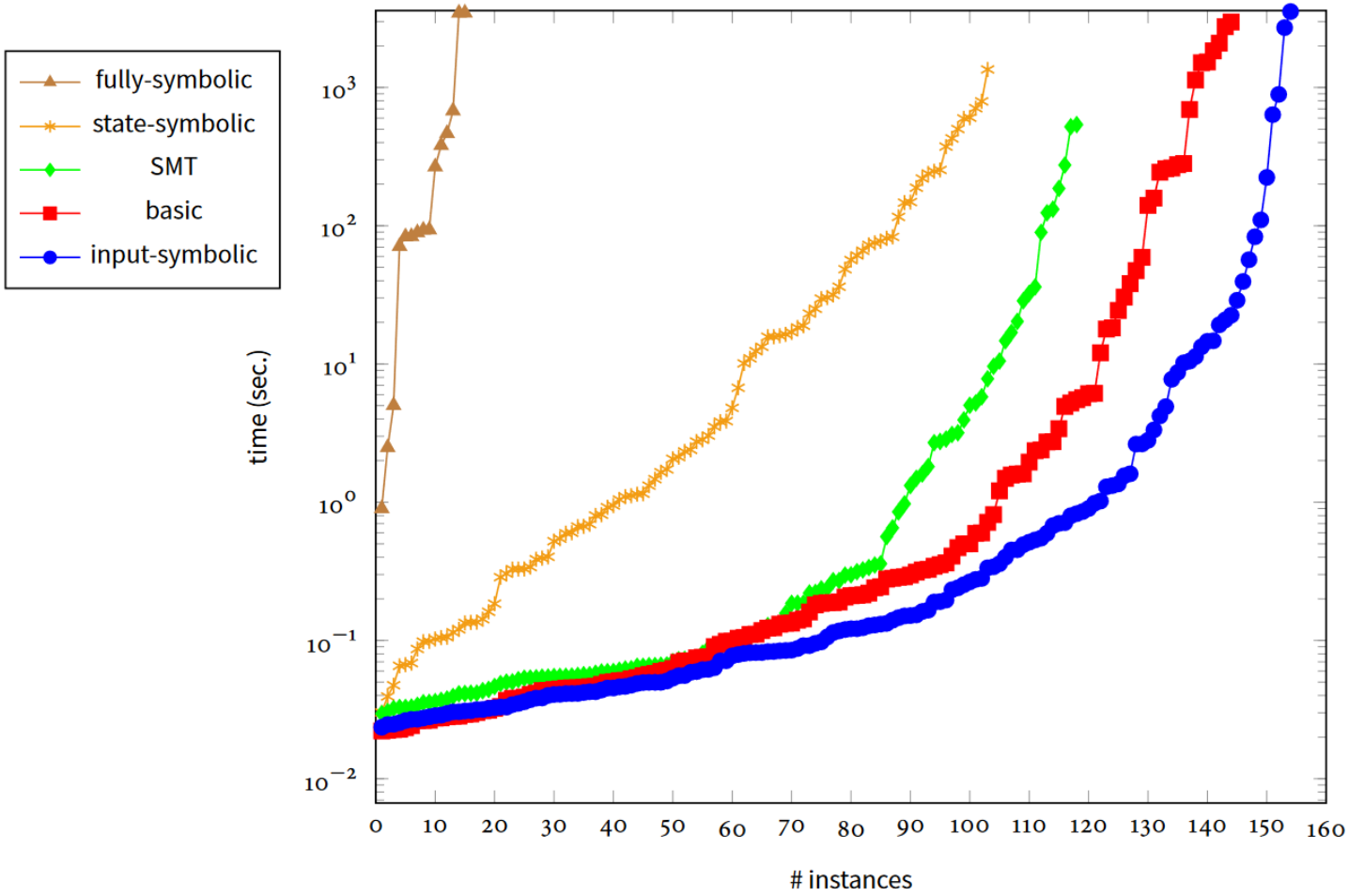
- DQBF: allows to specify which quantifiers another quantifier **depends on**. Like QBF encoding, but keep universal quantifiers for states of LTS (state-symbolic) and possibly also for states of the UCB (fully-symbolic).

Behaviour of different encodings

LTL realizability, SYNTCOMP 2016 experiment set



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For given LTL specification φ and $c \in \mathbb{N}$, the question whether an implementation with a c-bounded annotation exists can be answered by solving a **safety game**.

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Approach: expand co-Büchi automaton by adding states that count how often a rejecting state has been visited. If any state is visited more than c times, move to a rejecting sink state.

(other reductions are possible, and are used in practice)

Organization

What remains:

- Next Tuesday: Wrap-up lecture and presentation of competition results. In Seminar Room 15.
- Next Wednesday: Exam. Start at 14:30 (be there at 14:20) in Lecture Hall 3.