Introduction to Software Verification

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Lectures Material
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Lecture 8

Model Checking

- Emerging as an industrial standard tool for verification of hardware designs: Intel, IBM, Cadence, Mellanox, …
- Recently applied successfully also for software verification: SLAM (Microsoft), Java PathFinder and SPIN (NASA), BLAST (EPFL), CBMC (Oxford),…

Clarke, Emerson, and Sifakis won the 2007 Turing award for their contribution to Model Checking

Main Limitation of Model Checking:

The state explosion problem:

Model checking is efficient in time but suffers from high space requirements:

The number of states in the system model grows exponentially with

- the number of variables
- the number of components in the system

Solutions to the state-explosion problem

Symbolic model checking:

The model is represented symbolically

- BDD-based model checking
- SAT-based Bounded Model Checking (BMC)
- SAT-based Unbounded Model Checking

Other solutions to the state-explosion problem

Small models replace the full, concrete model:

- Abstraction
- Compositional verification
- Partial order reduction
- Symmetry

Symbolic (BDD-based) Model Checking for CTL

BDD-based Symbolic Model Checking

A solution to the state explosion problem:
BDD-based model checking

- **Binary Decision Diagrams (BDDs)** are used to represent the **model** and **sets of states**.

- It can handle systems with **hundreds** of Boolean variables.

Binary Decision Diagrams (BDDs)

- Data structure for representing Boolean functions

- Boolean function:

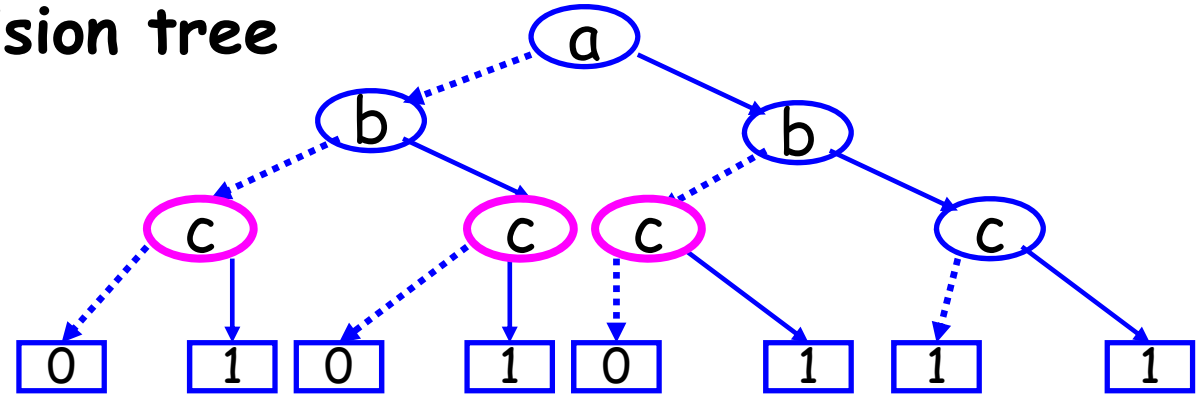
$$f: \{0,1\}^k \rightarrow \{0,1\}$$

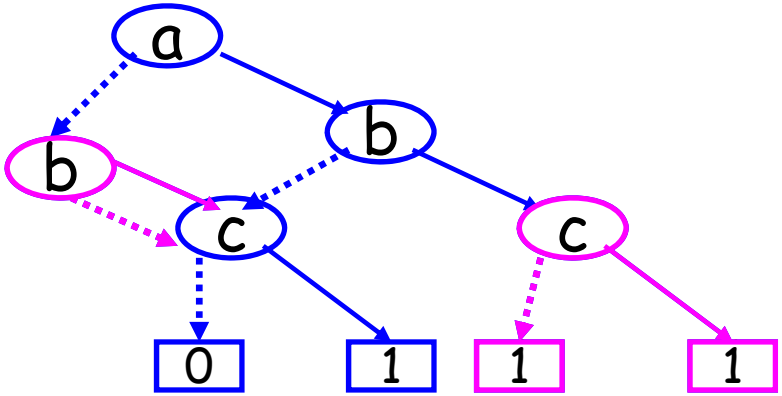
$$f(x_1, \ldots, x_k) = x_{k+1}$$

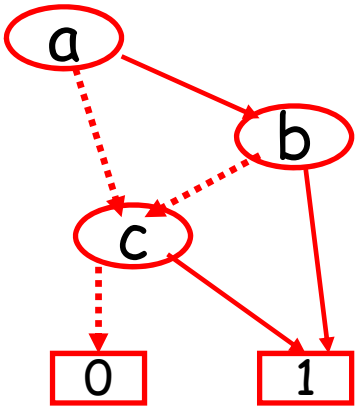
where $x_1, \ldots, x_k, x_{k+1} \in \{0,1\}$

BDD for f(a,b,c) = (a ∧ b) ∨ c

Decision tree







BDD

Binary Decision Diagrams (BDDs)

Advantages of BDDs:

- Often (but not always) **concise** in size
- **Canonical** representation
- Most **Boolean operations** can be performed on BDDs in **polynomial time** in the BDD size

BDDs in Model Checking

- Every set $A \subseteq U$ can be represented by its **characteristic function**

$$f_A(u) = \begin{cases} 1 & \text{if } u \in A \\ 0 & \text{if } u \notin A \end{cases}$$

- If the elements of U are encoded by sequences over $\{0,1\}^n$ then f_A is a **Boolean function** and can be represented by a BDD

- A Boolean function represents the set of all elements for which the function is 1

Representing a Model with BDDs

- Assume that **states** in model M are **encoded by $\{0,1\}^n$** and described by Boolean variables $v_1 \ldots v_n$
- S_f can be represented by a Boolean function (BDD) over $v_1 \ldots v_n$
- **R** (a set of pairs of states **(s,s')**) can be represented by a BDD over $v_1 \ldots v_n \; v_1' \ldots v_n'$

Example: Representing a Model with BDDs

$S = \{ s_1, s_2, s_3 \}$

$R = \{ (s_1,s_2), (s_2,s_2), (s_3,s_1) \}$

State encoding:

s_1: v_1v_2=00 s_2: v_1v_2=01 s_3: v_1v_2=11

For $A = \{s_1, s_2\}$ the Boolean formula representing A:

$f_A(v_1,v_2) = (\neg v_1 \wedge \neg v_2) \vee (\neg v_1 \wedge v_2) = \neg v_1$

Example: Representing a Model with BDDs

State encoding:

s_1: v_1v_2=00 s_2: v_1v_2=01 s_3: v_1v_2=11

For $A = \{s_1, s_2\}$ the Boolean formula representing A:

$f_A(v_1, v_2) = (\neg v_1 \wedge \neg v_2) \vee (\neg v_1 \wedge v_2) = \neg v_1$

f_A represents A if it gets the value 1 for every assignment which is an encoding of an element of A

$R = \{ (s_1,s_2), (s_2,s_2), (s_3,s_1) \}$

s_1: v_1v_2=00 s_2: v_1v_2=01 s_3: v_1v_2=11

$$f_R(v_1, v_2, v'_1, v'_2) =$$
$$(\neg v_1 \wedge \neg v_2 \wedge \neg v'_1 \wedge v'_2) \vee$$
$$(\neg v_1 \wedge v_2 \wedge \neg v'_1 \wedge v'_2) \vee$$
$$(v_1 \wedge v_2 \wedge \neg v'_1 \wedge \neg v'_2)$$

f_A and f_R can be represented by **BDDs**.

BDDs as a data structure

A BDD for a Boolean function $f(x_1, \ldots, x_k)$ is a directed acyclic graph (DAG) with a root and two types of nodes:

- Internal nodes v with fields
  - var(v) containing a variable name
  - Pointers low(v), high(v) to other nodes
- End nodes (leaves) v with field
  - value(v) $\in \{0,1\}$

Reduced, Ordered BDD (ROBDD)

To obtain a canonical representation, the following requirements are added:

- A variable appears at most once along every path from root to leaf
- The variables appear in the same order along every path from root to leaf
- The graph does not contain
  - isomorphic sub-graphs
  - Redundant nodes

Two graphs with root nodes u and v are isomorphic iff

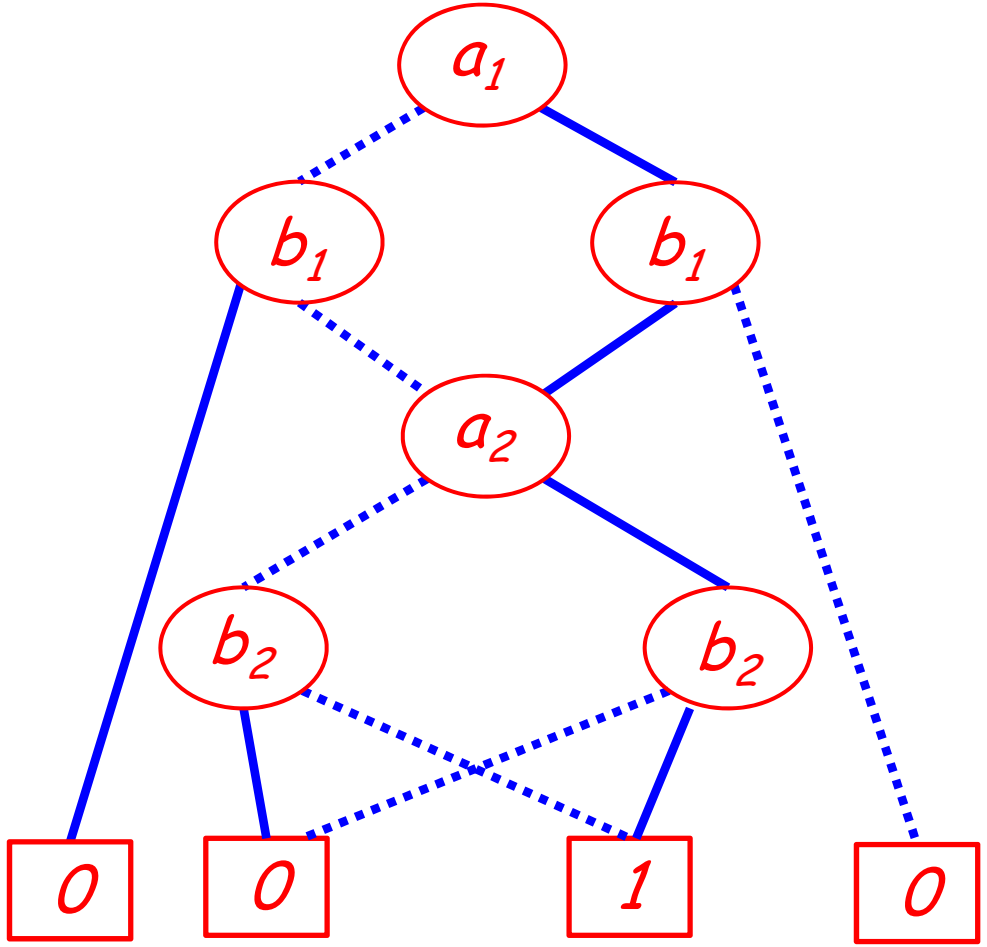
- If u and v are leaves then value(u)=value(v)
- If u and v are internal nodes then
  - var(u) = var(v)
  - low(u) and low(v) are isomorphic
  - high(u) and high(v) are isomorphic

A node v is redundant if low(v)=high(v)

Remark: From now on we will use BDD to denote ROBDD

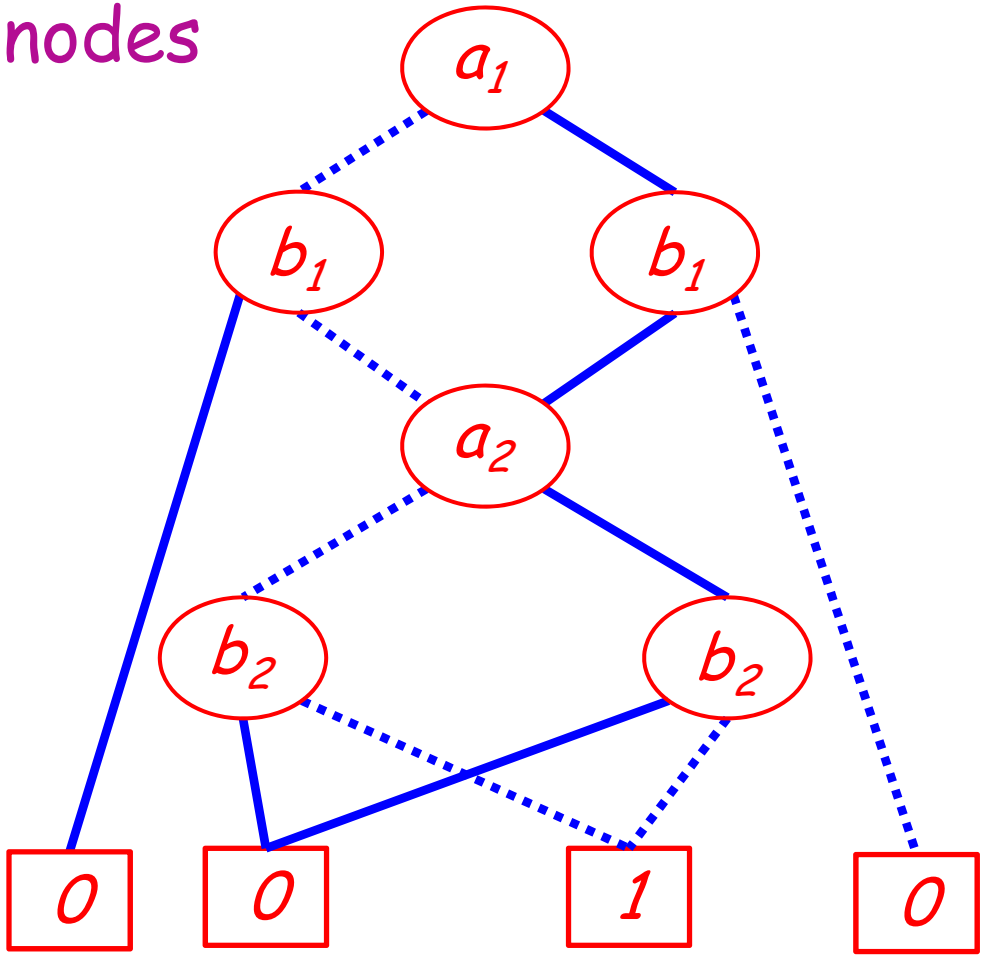
Order of Variables in the BDD

- $f(a_1,b_1,a_2,b_2) = (a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2)$
- Assume the variable order is: $a_1 < b_1 < a_2 < b_2$:



Order of Variables in the BDD

- In the general case: $a_1 < .... < a_n < b_1 < .... < b_n$
- $f(a_1, b_1, ..., a_n, b_n) = (a_1 \leftrightarrow b_1) \wedge ... \wedge (a_n \leftrightarrow b_n)$
- BDD with 3n+2 nodes



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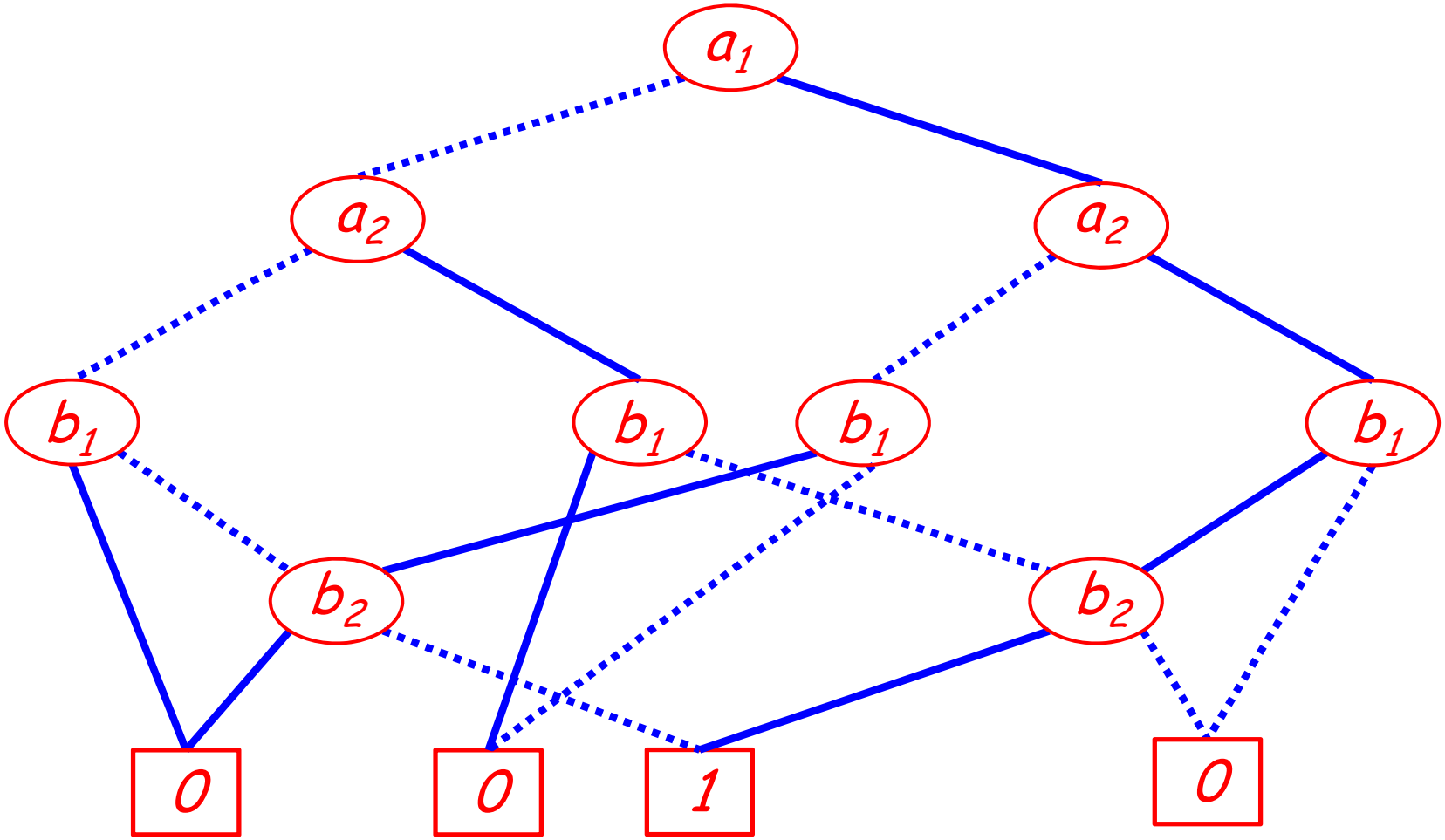


Order of Variables in the BDD

- In the general case: $a_1 < \ldots < a_n < b_1 < \ldots < b_n$:
- $f(a_1, b_1, a_2, b_2) = (a_1 \leftrightarrow b_1) \wedge \ldots \wedge (a_n \leftrightarrow b_n)$
- From a_1 to b_1 (including b_1) – full binary tree of depth n+1. Total of $2^{n+1}-1$ nodes
- Every b_1 remembers a specific assignment to $a_1,\ldots,a_n$. Thus 2^n nodes of b_1 (for all possible assignments of $a_1,\ldots,a_n$)
- Similarly, 2^{n-1} nodes of b_2 (for possible assignments of $a_2,\ldots,a_n$)
- .....
- 2 nodes of b_n (for possible assignments of a_n)

Order of Variables in the BDD

- In the general case: $a_1 < .... < a_n < b_1 < .... < b_n$:
- $f(a_1, b_1, a_2, b_2) = (a_1 \leftrightarrow b_1) \wedge \ldots \wedge (a_n \leftrightarrow b_n)$

- Sum of geometric sequence:

  $S_n = a_1 * (q^n - 1)/(q-1)$ (where $a_n = a_1 * q^{n-1}$)

- $2 + ... + 2*2^{n-1} = [2*(2^{n-1}-1)]/(2-1) = 2^n - 2$
- Total number of nodes:

  $2 + (2^n - 2) + (2^{n+1} - 1) = 3*2^n - 1$

Order of Variables in the BDD

Conclusion:

- The order of variables can influence significantly the size of the BDD

- Finding an optimal variable order is a hard problem
  - Heuristics are used

Given a BDD – which Boolean function does it represent?

A BDD with root v represents the function $f_v(x_1,\ldots x_n)$ as follows:

- If v is a leaf then $f_v(x_1,\ldots x_n) = \text{value}(v)$
- If v is internal and var(v) = x_i then

$$f_v(x_1,\ldots x_n) = (\neg x_i \wedge f_{low(v)}(x_1,\ldots x_n)) \vee (x_i \wedge f_{high(v)}(x_1,\ldots x_n))$$

Example

- $var(v_1)=a$, $var(v_2)=b$, $var(v_3)=c$
- $value(v_4) = 0$, $Value(v_5) = 1$
- $f_{v3}(a,b,c) = (\neg c \wedge f_{v4}(a,b,c)) \vee (c \wedge f_{v5}(a,b,c))$

  $= (\neg c \wedge 0) \vee (c \wedge 1) = c$
- $f_{v2}(a,b,c) = (\neg b \wedge f_{v3}(a,b,c)) \vee (b \wedge f_{v5}(a,b,c))$

  $= (\neg b \wedge c) \vee (b \wedge 1) = (b \vee c)$

- $f_{v1}(a,b,c) = (\neg a \wedge f_{v3}(a,b,c)) \vee (a \wedge f_{v2}(a,b,c))$

  $= (\neg a \wedge c) \vee (a \wedge (b \vee c)) =$

  $= (c \vee (a \wedge b))$

The BDD represents many functions, all equivalent to each other

Advantage of BDDs (revisited)

- Often (but not always) **concise** in size
- **Canonical** representation for a given variable ordering
  - Easy to check equivalence between two functions
- A function depends exactly on all variables that appear in its BDD
- Most **Boolean operations** can be performed on BDDs in **polynomial time** in the BDD size

Operations on BDDs

Operations on BDDs - Reduce

Reduce

Given an unreduced BDD:

- Eliminate isomorphic sub-graphs:
  - Eliminate duplicated end nodes
  - Eliminate duplicated internal nodes
- Eliminate redundant nodes

Reduce works bottom-up in linear time in the BDD size

Important remark:

BDD for a complex function is built bottom-up starting from small sub-functions to larger ones

We do not build a full decision tree and then reduce

Operations on BDDs - Restrict

Restrict

Given a BDD for $f(x_1,\ldots x_n)$, build a BDD for

$$f|_{x_i=b}(x_1,\ldots x_n) = f(x_1,\ldots,x_{i-1},b,x_{i+1},\ldots x_n)$$

Example:

$$f(x_1,x_2,x_3,x_4) = (x_1 \wedge x_2) \vee (x_3 \wedge x_4)$$

$$f|_{x_2=0}(x_1,x_2,x_3,x_4) = (x_1 \wedge 0) \vee (x_3 \wedge x_4) = (x_3 \wedge x_4)$$

Operations on BDDs - Restrict

Given a BDD A for $f(x_1,\ldots x_n)$, build a BDD for

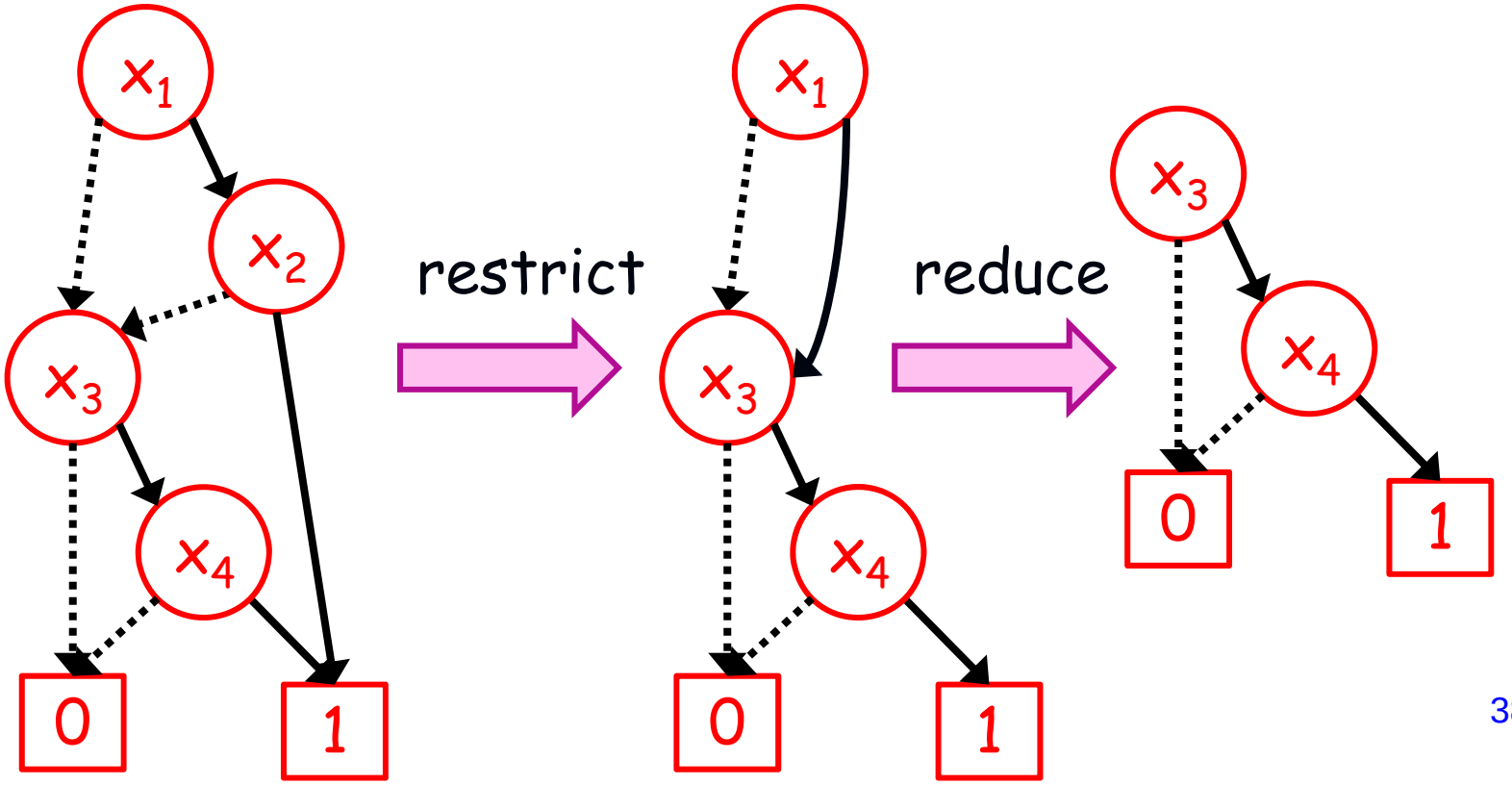
$$f|_{xi=b}(x_1,\ldots x_n) = f(x_1,\ldots,x_{i-1},b,x_{i+1},\ldots x_n)$$

- Traverse A from root to leaves
- For every node v with $var(v)=x_i$
  - Eliminate v from B
  - Replace edges to v by edges to low(v), if b=0 and to high(v), if b=1
- Run Reduce

Example: Restrict

$f(x_1,x_2,x_3,x_4) = (x_1 \wedge x_2) \vee (x_3 \wedge x_4)$

$f|_{x2=0}(x_1,x_2,x_3,x_4) = (x_1 \wedge 0) \vee (x_3 \wedge x_4) = (x_3 \wedge x_4)$



Operations on BDDs - Apply

- Gets two BDDs, representing functions **f** and **f'** and an operation ***
  - **Over the same variable ordering**
- Returns the BDD representing **f*f'**
- * can be any of 16 binary operations on two Boolean functions

16 binary operations

f	f'	f_1	f_2	f_3 … f_{15}	f_{16}
0	0	0	0	0 … 0	1
0	1	0	0	0 … 1	1
1	0	0	0	1 … 1	1
1	1	0	1	0 … 1	1
		const 0	AND	? OR	const 1

Operations on BDDs - Apply

- Shannon expansion
  for every Boolean function f and a variable x:

$$f = (\neg x \wedge f|_{x=0}) \vee (x \wedge f|_{x=1})$$

Notation:

- v,v' are the **roots** of f,f', respectively
- If v,v' are **not end nodes** then var(v)=x, var(v')=x'

Operations on BDDs - Apply

Computing f*f′ :

- Case 1: v and v′ are end nodes

  f*f′ = value(v) * value(v′)

- The BDD for f*f′

  consists of one leaf v″ with

  value(v″) = value(v) * value(v′)

This is the only case where * is taken into account

Operations on BDDs - Apply

Computing f*f′ :

- Case 2: x = x′
- Use Shannon expansion:

$$f*f' = (\neg x \wedge (f|_{x=0} * f'|_{x=0})) \vee (x \wedge (f|_{x=1} * f'|_{x=1}))$$

- Two simpler sub-problems to solve
  - Each depends on one less variable

End of lecture
5.12.2017

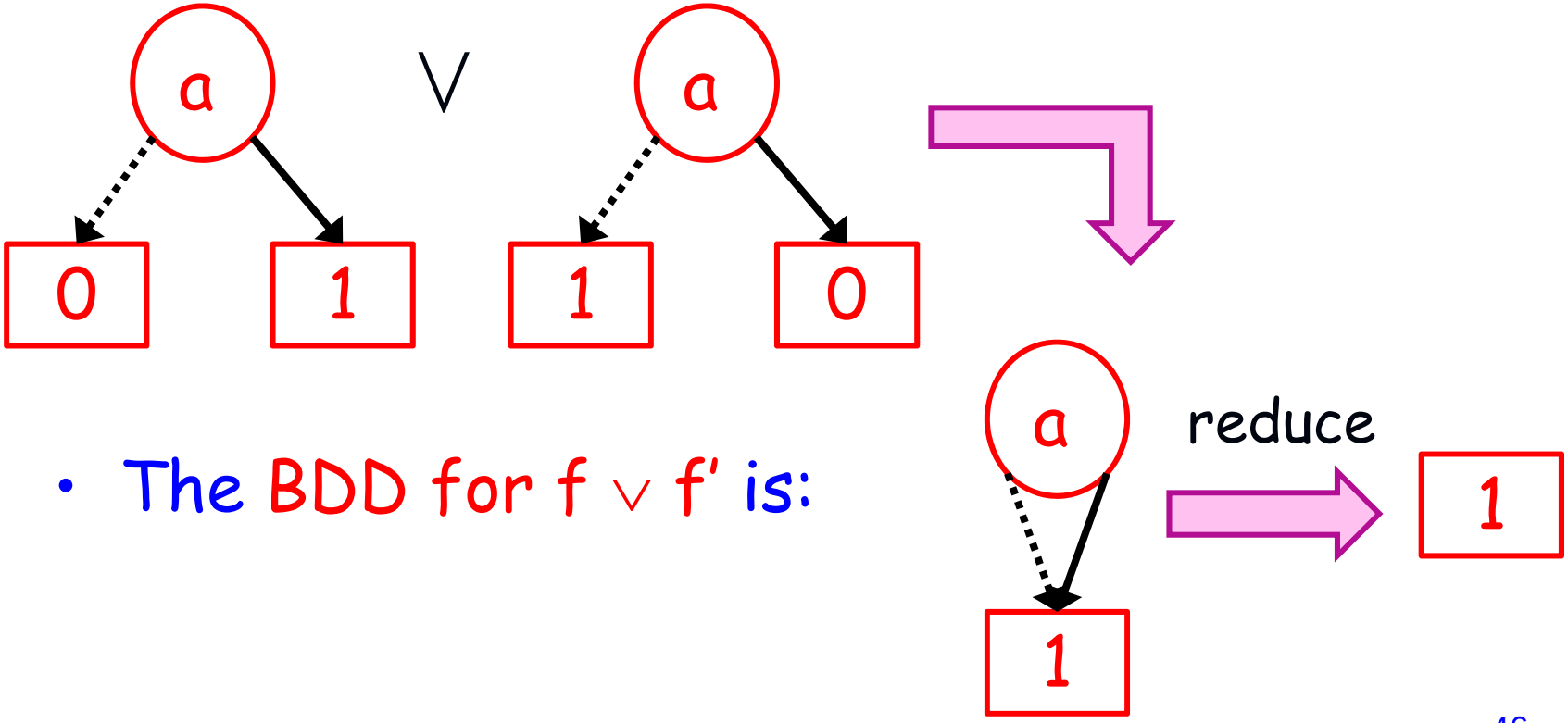
Operations on BDDs - Apply

Computing f*f′ :

- Case 2: x = x′
- The BDD for f*f′
- Root: a new node v″
  - var(v″) = x
  - low(v″) points to the root of the BDD for $(f|_{x=0} * f'|_{x=0})$
  - high(v″) points to the root of the BDD for $(f|_{x=1} * f'|_{x=1})$

Example

- f(a) = a, f'(a) = ¬a, * is ∨



- The BDD for f ∨ f' is: